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ANGULAR DISTRIBUTIONS FOR ELASTIC AND INELASTIC SCATTERING OF 40-MeV ALPHA PARTICLES BY CARBON, NITROGEN, OXYGEN, AND ARGON

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Received 9 March 1959

Abstract: Angular distributions for elastic and inelastic scattering of 40-MeV alpha particles by carbon, nitrogen, oxygen, and argon were obtained. Gas targets were used. Angles of observation ranged from 7° to 130° (laboratory) for carbon and over lesser intervals for the other elements. Elastic scattering angular distributions are characterized by sharply defined diffraction-like structure at the forward angles with maxima and minima spaced very regularly in sin $\frac{1}{2}\theta$ (c.m.), deep minima near 90° (c.m.), and appreciable cross sections at angles beyond 90°. A black nucleus diffraction interpretation gives reasonable absolute cross sections for the forward maxima and good agreement with observed locations of maxima and minima. Inelastic scattering angular distributions were obtained for C^{12} (Q = −4.43, −7.65 and −9.61 MeV); for O^{16} [Q = −6.06 and −6.14 MeV (unresolved), −6.91 and −7.12 MeV (unresolved)]; and A^{40} (Q = −1.462 MeV). Direct interaction analysis verifies previously assigned spins and parities and indicates an assignment of 0^+ or 2^+ to the 9.61-MeV level of C^{12}. Integrated cross sections for inelastic scattering to individual levels may be as high as 6 per cent of πR^2. Interaction radii determined from diffraction analysis of the elastic scattering data are (in units of 10^{-13} cm) 5.11 ± 0.20 for C^{12}, 5.37 ± 0.17 for N^{14}, 5.64 ± 0.29 for O^{16}, and 6.38 ± 0.32 for A^{40}. Interaction radii calculated from inelastic scattering data are consistently larger. Compound nucleus contributions to observed cross sections appear to be small or zero.

1. Introduction

The past several years have witnessed a rapid accumulation of experimental data on elastic and inelastic scattering of alpha particles in the energy region 15 to 50 MeV by elements ranging from beryllium to plutonium †††.

Analysis of the energy dependence [2]) of elastic scattering cross sections in terms of a sharp cut-off model has given a highly self-consistent picture of alpha-nucleus interaction radii over a wide range of medium and heavy elements. Angular distributions for elastic scattering have been analyzed in terms of sharp cut-off or modified sharp cut-off models and by optical model methods; recent optical model calculations have reproduced experimental angular distributions with good precision [3,4]). Although there seems to be some question about the uniqueness of the best values for real and imaginary

† Now at the University of Illinois, Urbana, Illinois.

†† Work supported in part by the U.S. Atomic Energy Commission.

††† A rather complete survey of recent experiments is given in ref. [1]).

nuclear potentials in the optical model analyses, information regarding the region near the nuclear surface is obtained [5,6]), and a short mean free path for alphas in nuclear matter is indicated. Mean interaction radii given by optical model calculations are generally somewhat smaller than the corresponding sharp cut-off radii; however, the radii are differently defined, and a close correspondence in terms of critical angular momenta can be made between optical model and sharp cut-off parameters [5,7]). Many angular distributions, particularly those for the light elements, exhibit very regular diffraction-like structure; in such cases the spacing of maxima and minima leads very simply, through use of Born approximation or simple diffraction theory results, to estimates of interaction radii. Application of these methods to the present results will be discussed below (sec. 4).

In the light elements, inelastic alpha-particle scattering resulting in excitation of low-lying nuclear levels is commonly observed; inelastic cross sections often exceed the corresponding elastic cross sections, especially at large angles. Angular distributions for particular inelastic alpha-particle groups generally show strong forward peaking, with oscillatory structure extending to fairly large angles (sometimes 90° or beyond). Interpretation of inelastic angular distributions in terms of direct interaction models [8–10]) yields interaction radii and sometimes provides a means of assigning spins and parities to excited states. In addition, the fact that certain cross sections for inelastic scattering are very appreciable at large angles may be used to infer that the corresponding excited states are characterized by collective excitations [9]); otherwise the large momentum transfer from the scattered alpha particle would lead to break-up of the nucleus.

A paradox appears to exist in the successful interpretation of alpha particle elastic scattering data in terms of optical model calculations in which a large attenuation of alpha particles in nuclear matter appears — or, similarly, by "black nucleus" diffraction approximations — on the one hand, as opposed to the apparently successful interpretation of inelastic scattering data in terms of simple direct interaction models which assume negligible distortion of incident and scattered plane waves, on the other †. Blair [13]) has recently suggested that a way out of the dilemma is provided by a diffraction inelastic scattering mechanism for alpha particles similar to that introduced by Drozdov [14]) to discuss inelastic neutron scattering from ellipsoidally deformed nuclei. However, the present inelastic scattering results have been analyzed in terms of previous simple direct interaction

† Born approximation direct reaction calculations which make use of distorted wave functions have been applied to the inelastic scattering of protons [11]). Although this method has not yet been applied to the inelastic scattering of alpha particles, there are strong indications that when large momentum transfer and appreciable nuclear penetration are involved the sharp oscillatory behaviour of the predicted angular distributions is lost [12]).

models (sec. 4); comparison with diffraction inelastic scattering predictions will be made at a future time.

The present angular distribution studies were made on He^4 †, C^{12}, N^{14}, O^{16} and A^{40}. They were undertaken as part of a continuing series [15–19]) of such experiments in the hope of obtaining information on alpha-nucleus interaction radii; on the transparency of nuclei to medium-energy alpha particles; on the validity of the Born approximation elastic scattering analysis; on spins and parities; on excitation characteristics of excited states of the target nuclei; and on the question of whether inelastic scattering to a particular excited state proceeds principally through interaction of the alpha with a single nucleon, with a group of nucleons, or with the entire target nucleus.

2. Apparatus and Experimental Procedure ††

2.1. CYCLOTRON EXTERNAL BEAM SYSTEM

The University of Washington 60-inch cyclotron accelerates alpha particles to energies of about 42 MeV with beam strengths of up to 100 μA. The external beam system consists principally of an adjustable slit just outside the cyclotron proper, a double-focusing uniform-field wedge magnet, a precision slit system, and a second double-focusing uniform-field wedge magnet which brings the beam to a focus in a scattering chamber located in a shielded room about 50 feet from the cyclotron. The high dispersion of the second magnet (about 50 keV/mm for 40-MeV alpha particles) makes precise energy definition at the target feasible when a tight collimating system is used. Energy-analyzed beams of current densities up to 6 $\mu A/cm^2$ are available. A more complete description of the system will be given elsewhere.

2.2. SCATTERING CHAMBER AND GAS TARGET CHAMBER

A 24-inch diameter scattering chamber contains the target assembly (foil holder or gas target chamber) and a particle detector. Target and detector can be rotated by means of remote-control selsyns. A cylindrical Faraday cup is mounted opposite the collimator system for the incoming beam, and a monitor counter is located at a fixed angle to the beam (sec. 2.5).

Most of the present experiments made use of the gas chamber-particle detector assembly shown in fig. 1. (Some of the work with carbon was done with thin foil targets conventionally mounted.) Fig. 2 shows a section view, in the plane of incident and scattered particles, of the gas chamber and scattered particle collimator cone. Laboratory angles from 12° to 168° are accessible. Gas pressure is controlled from outside the scattering chamber, which is evacuated.

† The work on He^4 will be the subject of a separate paper.

†† A more extensive discussion is presented in ref. [20]).

Incident alpha particles can be scattered by the aluminium foil entrance and exit windows or by the target gas. Only those scattered from a target gas volume defined by the incident beam collimator system and the scattered particle collimator cone components can reach the detector directly. The useful scattering volume is proportional to cosec θ, where θ is the laboratory scattering angle, provided that the axis of the beam collimator system, the axis of the scattered particle collimator, and the axis of rotation of the gas target chamber meet at one point for all angles. Careful checks, made both

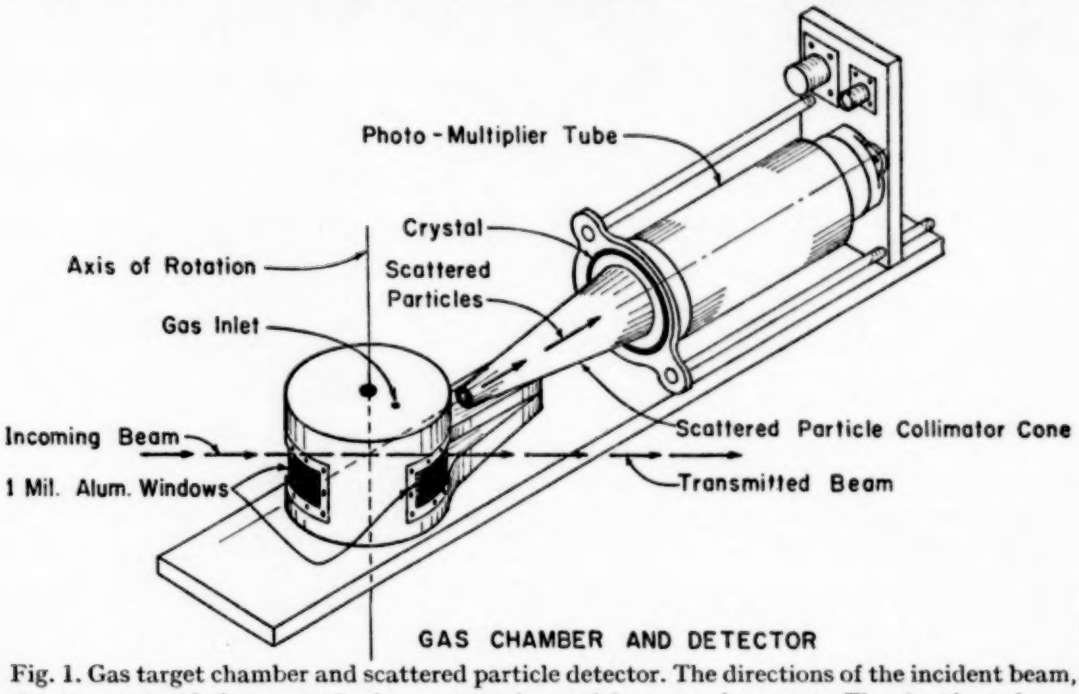

Fig. 1. Gas target chamber and scattered particle detector. The directions of the incident beam, the transmitted beam and the scattered particles can be seen. The incident beam passes through a one-mil aluminium window upon entering the chamber, and the transmitted beam passes through a similar window upon leaving the chamber. The scattered particles pass through a $\frac{1}{4}$-mil aluminized mylar window upon leaving the chamber. Only those particles scattered into a cone defined by the scattered particle collimators can impinge on the detector crystal. The photomultiplier tube, which is located just behind the crystal, is surrounded by a magnetic shield. Gas pressure is controlled from outside the scattering chamber, which is evacuated. (Connections to the gas supply are not shown.)

mechanically and with the aid of particle beams, verified the common intersection of all three axes within acceptable limits ($\approx \frac{1}{64}$ inch). The cosec θ dependence of the sensitive volume was verified by comparing the angular distributions for elastic scattering from solid and gas carbon targets (sec. 3). In the first twenty degrees (18°—38°) of overlap, the region in which cosec θ is changing most rapidly (a factor 2.0 in this interval), the two sharply diffraction-like angular distributions were brought into very satisfactory agreement — within limits of a normalization uncertainty of five to ten percent — by application of the cosec θ correction factor to the gas target data.

Angular uncertainties were not more than $\pm\frac{1}{2}°$. Tests of the angular calibration for the gas target system included a coincidence test in which helium gas served as the target. With an auxiliary counter placed at $-45.0°$, coincidences between the auxiliary counter and the gas chamber detector

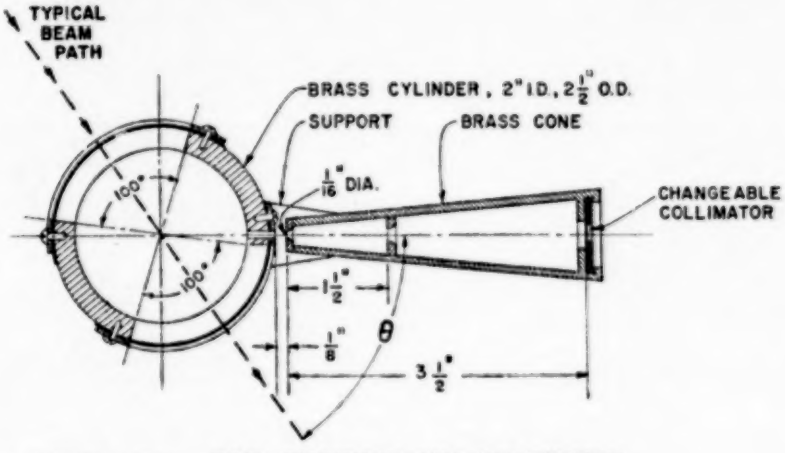

Fig. 2. Gas target chamber and scattered particle collimator cone: a section taken in the plane of incident and scattered particles. On the side of the cylinder, radially opposing each other, are two windows, each covering an angular aperture of 100°. At 1/16 inch from the edge of one window is a hole 3/32 inch in diameter. Oriented radially outward from this hole is a cone $3\frac{1}{2}$ inches in length. The cone contains a first collimator with a circular aperture 1/16 inch in diameter, a second circular collimator 5/32 inch in diameter located $1\frac{1}{2}$ inches from the narrow end, and a third and final collimator that can easily be changed. All collimators are chamfered, leaving only a thickness of 15 mils at the front face, in order to reduce edge scattering. Laboratory angles of observation θ in the interval 12° to 102° are covered with the beam incident from the side indicated in the figure; the interval 78° to 168° is covered by reversing the roles of entrance and exit windows.

were found to reach a maximum for $\theta = +44.8°$, which compares nicely with the expected angle of 45.0°.

2.3. TARGET MATERIALS

The first scattering experiments on C^{12} were done with solid targets. The one usually used was 2.1 mg/cm² polystyrene foil. A second type of solid target, an almost pure carbon foil prepared from a dag suspension in benzene, contained smaller amounts of impurities but was not sufficiently uniform for accurate estimate of the number of C^{12} atoms exposed to the beam. At backward angles the energy spread due to the thickness of either of the above types of targets was comparable to the energy separation between the different inelastic alpha groups. To obviate this difficulty a very thin and uniform target was made by dissolving polystyrene in benzene and spreading it on water. It was left there for several hours. The thin polystyrene film was then cut in the water to the right size, picked up from

the water, quickly Scotch-taped onto a frame, and only then allowed to dry out.

Four target gases were used: acetylene, nitrogen, oxygen and argon. They were the regular commerical gases. Chemical impurities were always less than $\frac{1}{2}$ %. The isotopic purity of the isotopes of interest (C^{12}, N^{14}, O^{16} and A^{40}) in these gases is 99 % or better. The pressure in the chamber was usually about 1 atmosphere, and was measured with the help of a mercury manometer to within about 0.1 %. Possible dependence of the observed differential cross sections upon the pressure (due to double scattering) was taken into account by making the same measurements at pressures of $\frac{1}{2}$ atmosphere and 1 atmosphere; no change in the measured cross section at a given angle was observed.

2.4. DETECTORS

The detector used for most of the experiments was a scintillation spectrometer apparatus consisting of a crystal and a Dumont 6291 photomultiplier tube (fig. 1). Three kinds of crystals were used: NaI(Tl), CsI(Tl), and plastics. The NaI cyrstals gave the best energy resolution but were hardest to handle because of their hygroscopic properties; frequent use was made of CsI crystals, which were nearly as good from the standpoint of resolution.

Effective discrimination against protons and other light particles was achieved by using crystals sanded to different thicknesses. Each crystal was used in an appropriate range of scattering angles such that the elastically scattered alpha particles would be stopped close to the second surface of the crystal but still within it, thus yielding a much larger pulse than an energetic proton or deuteron stopping in or passing through the crystal.

Pulses from the detector photomultiplier passed through conventional preamplifier and amplifier circuits to a forty-channel differential pulse-height analyzer, where the pulse spectrum was displayed. (Actually, one hundred twenty channels were available, of which any adjacent forty could be selected by base-line adjustment for use in a single count.) Accurate identification of inelastic alpha groups depended upon a well-established correspondence between energy and pulse height. Such a calibration for a typical CsI crystal-photomultiplier combination is shown in fig. 3. Typical pulse-height spectra will be presented in sec. 3.

2.5. BEAM CURRENT INTEGRATION

Charge collected on the Faraday cup was calculated from the potential difference, measured with a Keithley electrometer, across a calibrated capacitor connected to the Faraday cup and to ground.

A monitor counter was used to supplement and to check the consistency of the Faraday cup measurements. A CsI crystal-photomultiplier unit was mounted externally at an angle of 30° to the incident beam; it viewed the

target through a thin beryllium-copper window. Amplification and integral discrimination of monitor pulses provided a means of counting appropriate groups of alphas scattered by solid targets or, in gas target experiments, alphas scattered from the aluminium exit window foil (fig. 1). The monitor

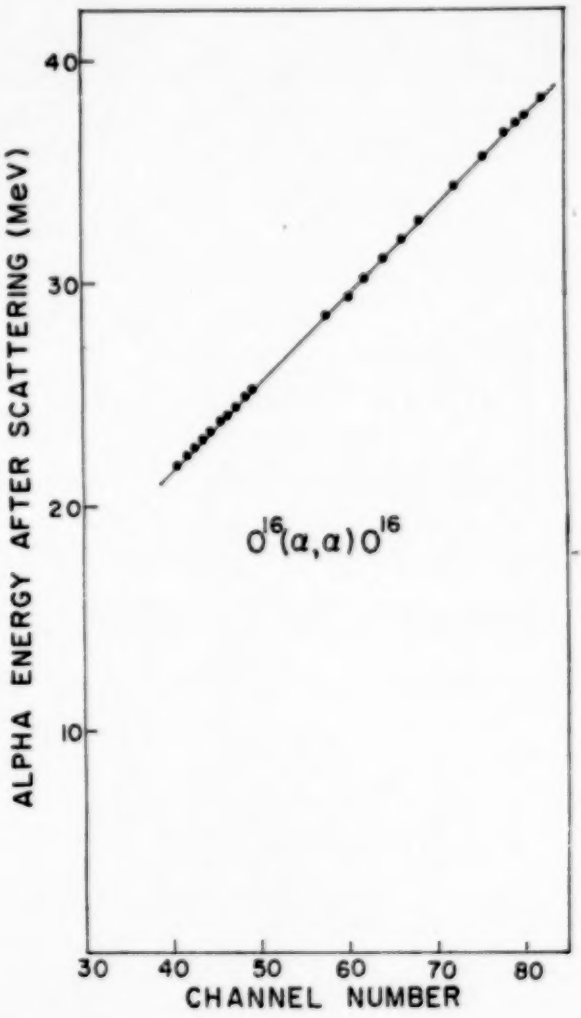

Fig. 3. Observed pulse height *versus* energy for alpha particles scattered elastically from O^{16}. The energy of the alphas is calculated kinematically for laboratory scattering angles between 20° and 100°, assuming an incident alpha energy of 40 MeV. The channel number for a given energy represents the position, in a multichannel analyzer display, of the pulse height group from the elastically scattered alphas.

was essential at very small angles in the carbon solid target experiments, where the beam to the Faraday cup was intercepted by the detector mount and the monitor provided the only method of integration. Faraday cup and monitor counter integrations were found to be mutually consistent within $\pm 3\,\%$ at worst.

2.6. BEAM ENERGY

Incident alpha-particle energies were determined from measurements of the mean range in aluminium of alpha particles scattered elastically from a thin gold foil target. Recent range-energy data [21]) were used. Since certain differential cross sections (notably $C^{12}(\alpha, \alpha')C^{12*}$, $Q = -4.43$ MeV) were found to be very energy-sensitive, careful regulation of adjustable cyclotron and external beam system parameters was necessary. Beam energy stability to ± 0.1 MeV was possible, though shifts of as much as 0.5 MeV were sometimes observed after cyclotron adjustments. Fluctuations in beam energy were not severe enough to be the limiting factor in the energy resolution obtainable in the present experiments (sec. 3.2).

3. Experimental Results

3.1. SCATTERING BY C^{12} AT 41 MeV (SOLID TARGET)

Angular Distributions

Observed laboratory angular distributions for elastic scattering and inelastic scattering ($Q = -4.43$ MeV) are shown in fig. 4 †. A calculated angular distribution for point charge Coulomb (Rutherford) scattering is also given for comparison. Both elastic and inelastic angular distributions exhibit a pronounced oscillatory structure, with the average value of the elastic differential cross section dropping much more rapidly with increasing angle than the inelastic. The inelastic cross section holds up well at large angles, and is consistently much greater than the elastic at angles beyond about 60°.

Absolute Cross Sections

Although it was not the intent of the present experiments to make accurate measurements of absolute cross sections, it was possible to establish a reasonably good absolute scale. The angular distribution of alpha particles scattered elastically from gold follows very closely the Rutherford dependdence between 15° and 22° [21]). Making use of this fact, the effective acceptance solid angle of the detector used in the solid target experiments was determined by observing elastic scattering from a thin (0.0003-inch) gold foil. Absolute differential cross sections in the solid carbon target experiment could thus be calculated, and gas target carbon differential cross sections normalized to them. A means of calculating absolute cross sections for other gas target experiments was also thereby established. A probable error of ± 10 % in the scale of absolute cross sections, due principally to the necessity

† Excitation energies and spin and parity assignments for C^{12}, N^{14} and O^{16} are taken from ref. [22]). Level assignments for A^{40} were taken from ref. [23]).

for this procedure, is assumed for the solid target experiments. The uncertainty is larger for the gas target experiments (see below).

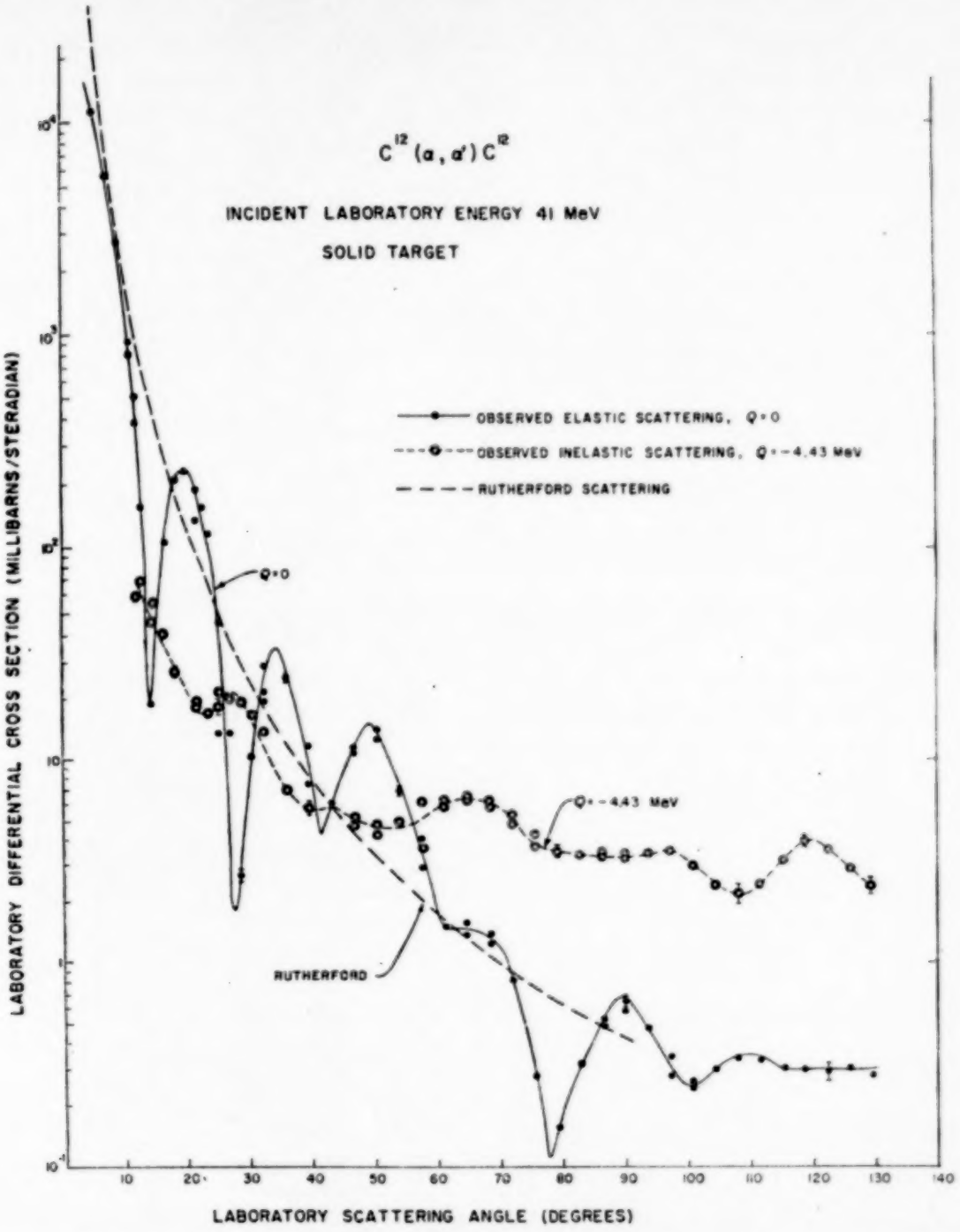

Fig. 4. Laboratory angular distributions for alpha particles scattered by C^{12} (solid target) at incident laboratory energy 41 MeV. Elastic scattering ($Q = 0$) and inelastic scattering ($Q = -4.43$ MeV) differential cross sections are shown. A calculated Rutherford angular distribution is also given. The probable error in the scale of absolute differential cross section is ± 10 %. Representative statistical errors (standard deviation) are shown.

3.2. SCATTERING BY C^{12} AT 40 MeV (GAS TARGET)

Angular Distributions

Observed laboratory angular distributions for elastic scattering and inelastic scattering ($Q = -4.43$, -7.65 and -9.61 MeV) are shown in fig. 5. The target material is acetylene.

With the same incident energy as before, the effective incident laboratory energy is reduced to 40 MeV for this case, since the alpha particles lose about 1 MeV in passing through the aluminium entrance window and approximately 1 inch of gas at atmospheric pressure.

Pulse Height Spectra and Energy Resolution

The groups of scattered alpha particles corresponding to different states of the residual C^{12} nucleus were identified at each angle from the detector pulse-height spectrum. Fig. 6 shows the pulse-height spectrum at laboratory scattering angle 27°. The observed positions of the $Q = -7.65$ MeV and $Q = -9.61$ MeV peaks agree within one channel with the positions predicted from the observed separation of the $Q = 0$ and $Q = -4.43$ MeV peaks. The $Q = -7.65$ MeV and $Q = -9.61$ MeV peaks are superimposed on a continuous background due principally to the breakup of the C^{12} nucleus into alpha particles [25, 26]). Subtraction of this continuum introduces uncertainties which increase rapidly with scattering angle.

The energy resolution of the detector system is easily estimated from the data of fig. 6. The full width of the elastic peak at half maximum counting rate is 2.0 channels; translation of this peak width into energy indicates an energy resolution of 1.8 %. The elastic peak width at one-tenth maximum counting rate is 4.1 % in energy, and the peak-to-valley ratio for the first two peaks and valleys is about 50/1. These figures characterize favourable, but by no means unrepresentative, operating conditions. At very large angles target thickness effects broaden the peaks, but the principal limitation appears to be the crystal-photomultiplier performance.

Normalization and Energy Dependence

The scale of absolute differential cross section was established for the C^{12} gas target data by normalization of the angular distribution for elastic scattering against that obtained with the solid target. Test runs made at several energies chosen about 1 MeV apart demonstrated that at angles below about 60° the peak and valley positions and peak heights in the elastic angular distribution were not significantly sensitive to small energy changes. The peaks at (approximately) 21°, 35° and 50° were therefore used to establish the best match between gas target (40 MeV) and solid target (41 MeV) data. [Diffraction scattering calculations would lead one to expect that forward peak magnitudes might be in direct proportion to

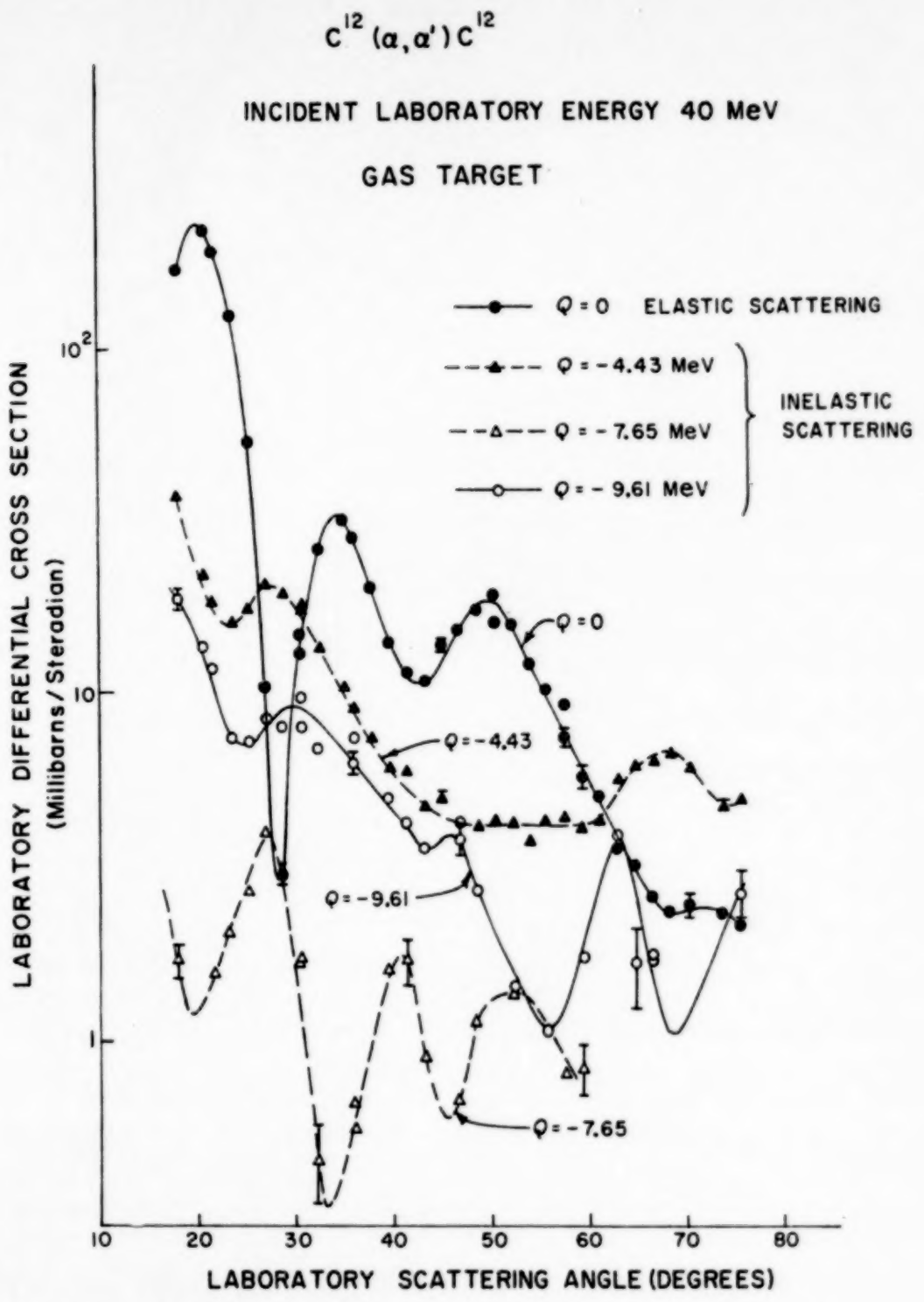

Fig. 5. Laboratory angular distributions for alpha particles scattered by C^{12} (gas target) at incident laboratory energy 40 MeV. Differential cross sections are plotted for elastic scattering ($Q = 0$) and for inelastic scattering with $Q = -4.43$, -7.65 and -9.61 MeV. The probable error in the scale of absolute differential cross section is ± 15 %. Representative errors shown on plotted points are from counting statistics ($Q = 0$, -4.43 MeV curves) or from counting statistics and alpha-particle continuum subtraction ($Q = -7.65$, -9.61 MeV curves).

incident energy (sec. 4); thus a 2—3 % advantage should be given to the solid target peaks. In the compromise best match, the 21° and 35° peaks for 41 MeV were slightly high, the 50° peak slightly low.]

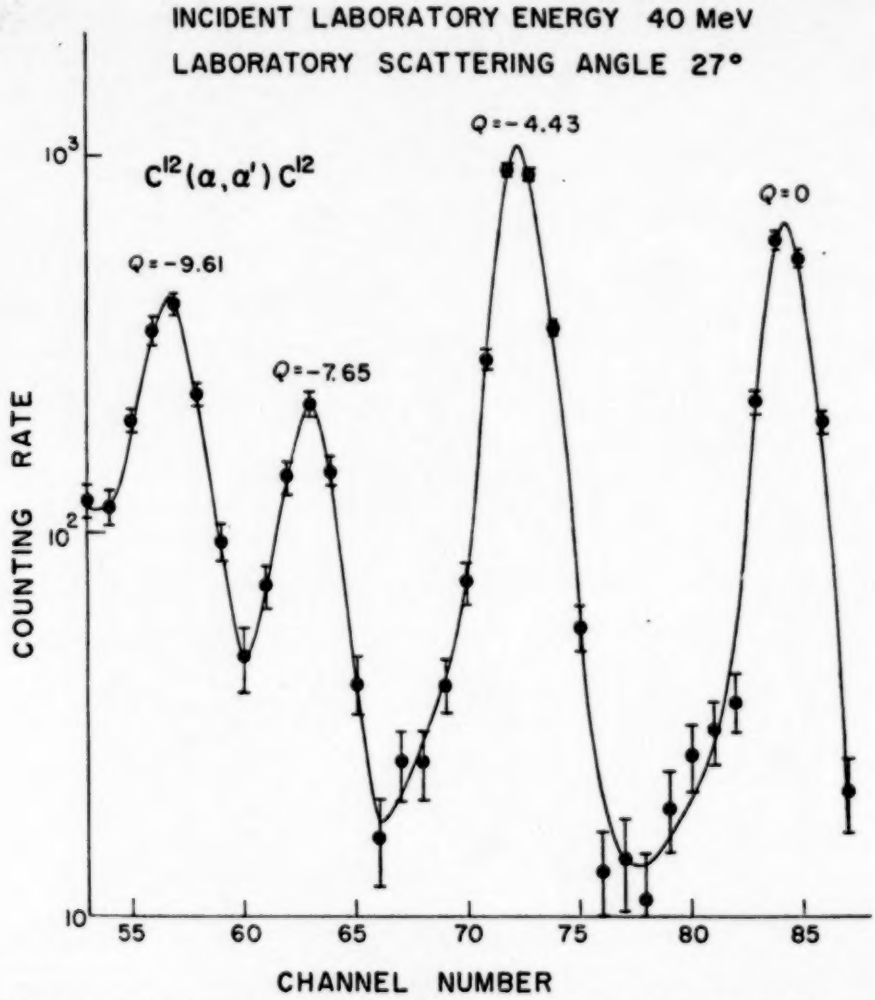

Fig. 6. Pulse-height spectrum for C^{12} (α, α')C^{12} at laboratory scattering angle 27° and incident laboratory energy 40 MeV. The counting rate is expressed in counts per 7×10^4 monitor counts. Pulse height is expressed in channel number on a 120-channel analyzer (sec. 2.4). The observed positions of the $Q = -7.65$ and $Q = -9.61$ MeV peaks agree within one channel with the positions predicted from the observed separation of the $Q = 0$ and $Q = -4.43$ MeV peaks. The $Q = -7.65$ MeV and $Q = -9.61$ MeV peaks are superimposed on a continuous background due principally to the breakup of the C^{12} nucleus into alpha particles. Energy resolution is about 1.8 % (full width at half maximum) for the elastic peak ($Q = 0$).

Other contributions to the overall uncertainty in absolute cross section are those due to uncertainties in density of target atoms (determined from temperature and pressure in the gas target experiments), laboratory scattering angle (sec. 2.2), and charge collected. All are considerably smaller than the normalization uncertainty. An overall probable error of ± 15 % is assigned to all scales of absolute differential cross section for the gas target experiments on C^{12}, N^{14} and O^{16}, and ± 20 % to the A^{40} scale.

Although the diffraction-like structure for elastic scattering at small angles does not change rapidly with energy, a close comparison of data at 41 MeV and data at 40 MeV will show that the latter exhibit a partial filling

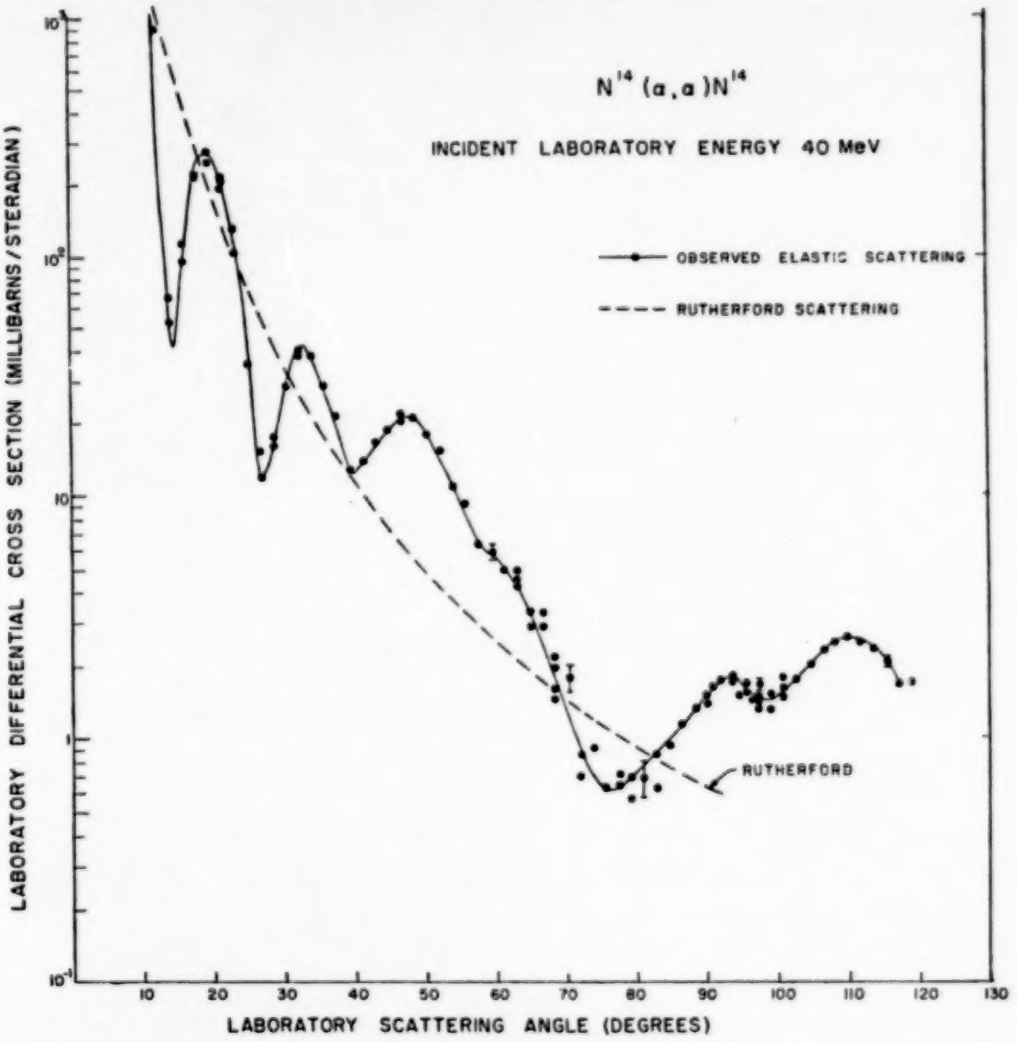

Fig. 7. Laboratory angular distribution for alpha particles elastically scattered by N^{14} at incident laboratory energy 40 MeV. A Rutherford angular distribution is also shown. The probable error in the scale of absolute differential cross section is ± 15 %. Representative statistical errors (standard deviation) are shown.

of the valley just above 40° and appreciably larger cross sections at angles beyond about 55°. The inelastic angular distribution ($Q = -4.43$ MeV) in the region 35° to 70° was found by repeated observation to be quite sensitive to small energy changes, with higher cross sections corresponding on the average to higher energies.

Other Investigations of Scattering of Alpha Particles by C^{12}

Other investigations made in this laboratory at approximately the same bombarding energy include (a) a more detailed study of the inelastic

($Q = -7.65$ MeV) angular distribution and the subsequent breakup of the C^{12} nucleus [26]), (b) a measurement of the alpha-gamma angular correlation [19]) for inelastic scattering to the 4.43-MeV state with subsequent deexcitation

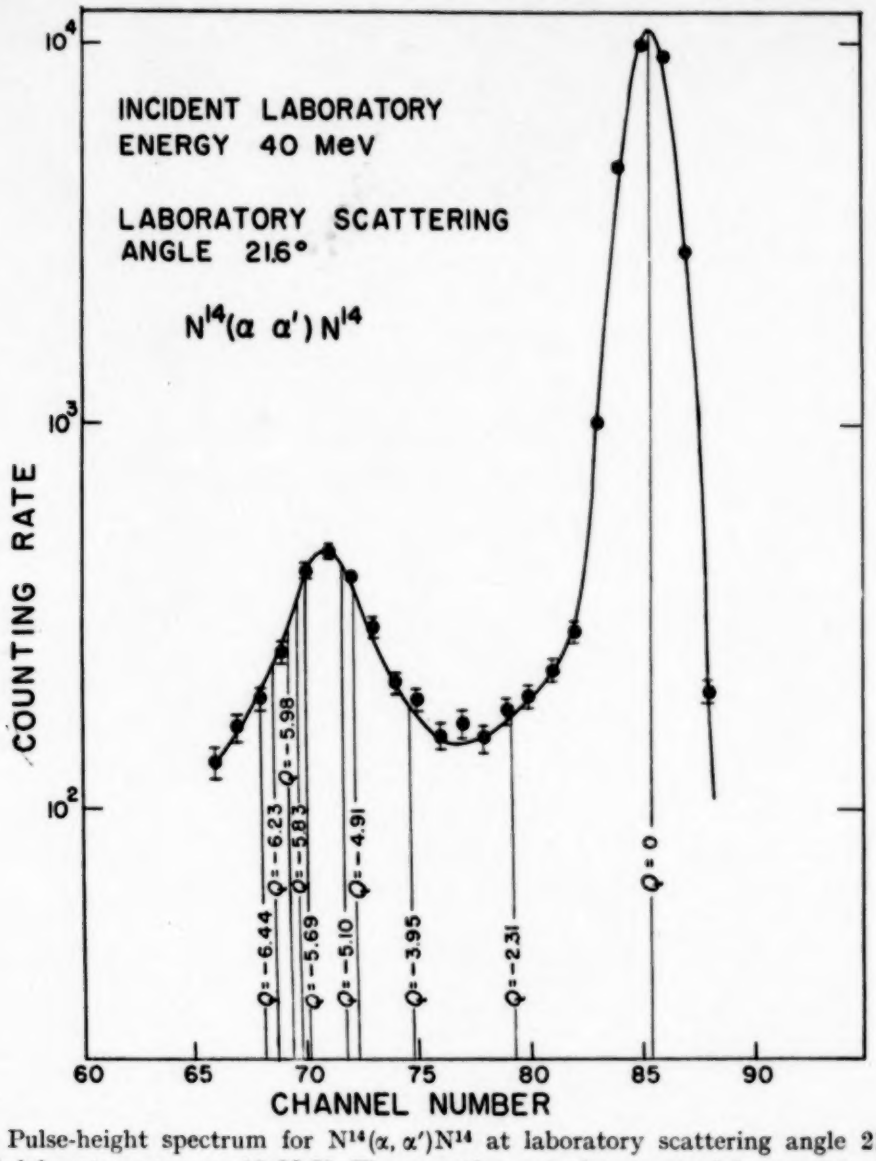

Fig. 8. Pulse-height spectrum for $N^{14}(\alpha, \alpha')N^{14}$ at laboratory scattering angle 21.6° and incident laboratory energy 40 MeV. The counting rate is expressed in counts per 2×10^4 monitor counts. The energies of the inelastically scattered alphas which leave the N^{14} nucleus in low-lying excited states are calculated kinematically, while the channels in which the corresponding peaks are expected to occur are predicted from a pulse height *versus* energy curve such as that shown in fig. 3 for $O^{16}(\alpha, \alpha)O^{16}$.

by gamma emission. Investigations made at other energies are described in refs. [27–30]). Reported elastic scattering results [31]) at 40 MeV over a limited angular region show good agreement with the present results with respect to location of maxima and minima but somewhat higher cross sections at the second and third maxima. Large angle data are not available. Unpublished inelastic scattering results [32]) show qualitative similarity at small angles.

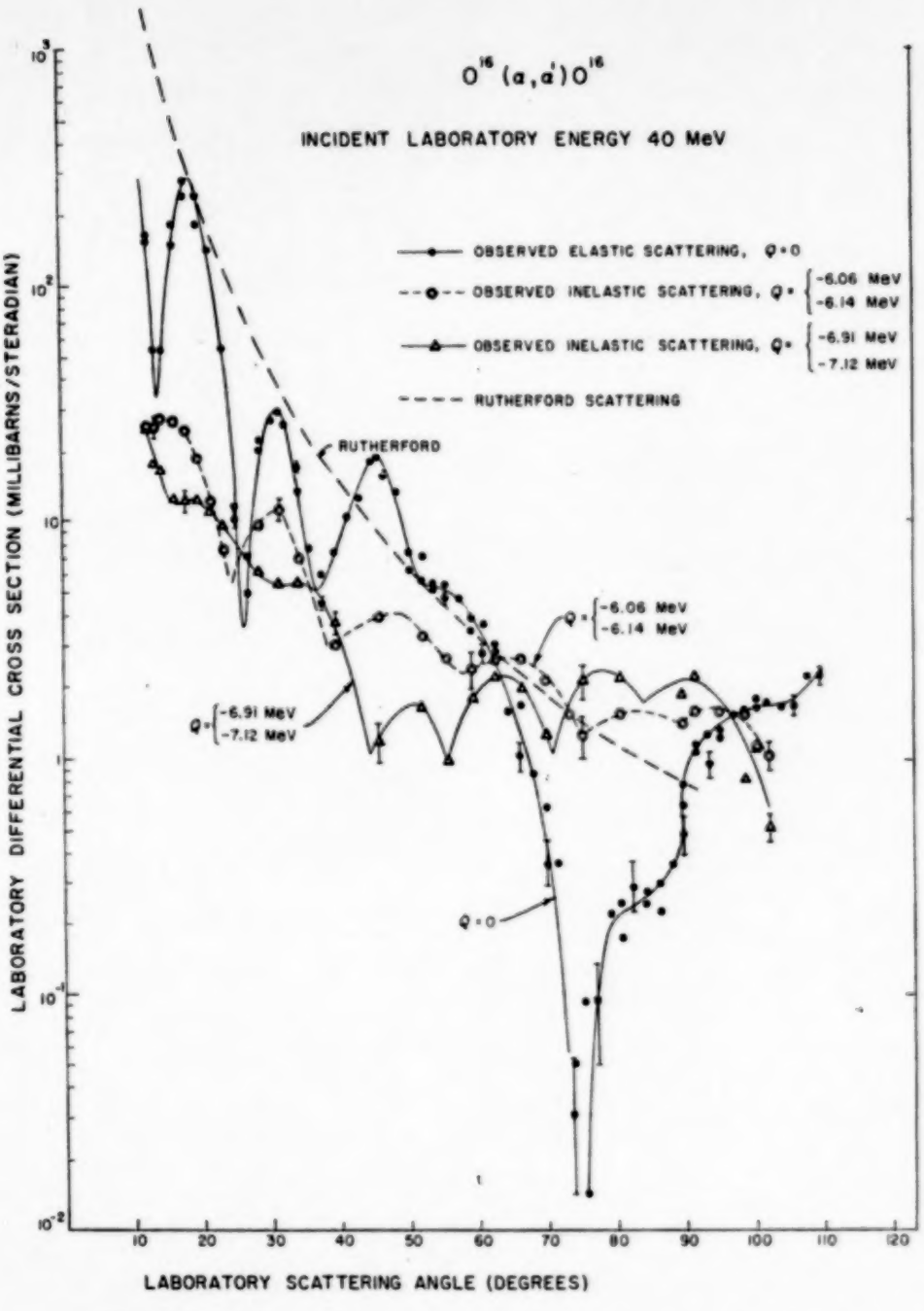
$O^{16}(a,a')O^{16}$
INCIDENT LABORATORY ENERGY 40 MeV
OBSERVED ELASTIC SCATTERING, Q=0
OBSERVED INELASTIC SCATTERING, Q= -6.06 MeV, -6.14 MeV
OBSERVED INELASTIC SCATTERING, Q= -6.91 MeV, -7.12 MeV
RUTHERFORD SCATTERING
RUTHERFORD
Q= -6.06 MeV, -6.14 MeV
Q= -6.91 MeV, -7.12 MeV
Q=0
LABORATORY DIFFERENTIAL CROSS SECTION (MILLIBARNS/STERADIAN)
LABORATORY SCATTERING ANGLE (DEGREES)

Fig. 9. Laboratory angular distributions for alpha particles scattered elastically and inelastically by O^{16} at incident laboratory energy 40 MeV. The two levels of the doublet $Q = -6.06$, -6.14 MeV and those of the doublet $Q = -6.91$, -7.12 MeV are not resolved, and the angular distributions are plotted as if each doublet were a single level. The oscillatory structure of the inelastic angular distributions is emphasized by approximate curves which are not smooth. The representative errors shown for the inelastic angular distributions include contributions from counting statistics, separation of the two doublets from each other, and subtraction of a background continuum (fig. 10). The probable error in the scale of absolute differential cross section is ± 15 %.

3.3. SCATTERING AT 40 MeV BY N^{14}, O^{16} AND A^{40} (GAS TARGETS)

The observed angular distribution for elastic scattering by N^{14} is presented in fig. 7. Counter resolution during the N^{14} runs was inadequate to permit separation of individual alpha-particle groups from inelastic scattering.

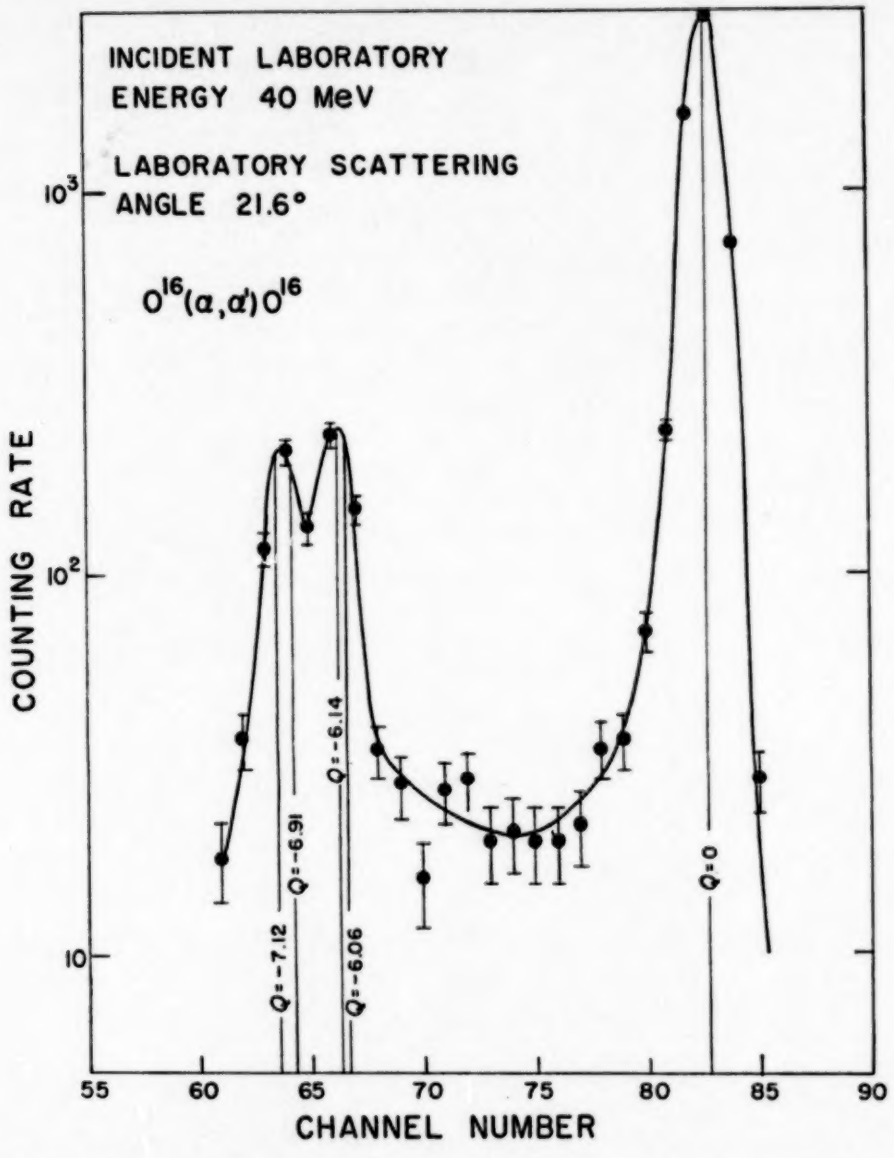

Fig. 10. Pulse-height spectrum for $O^{16}(\alpha, \alpha')O^{16}$ at laboratory scattering angle 21.6° and incident laboratory energy 40 MeV. The counting rate is expressed in counts per 2.5×10^4 monitor counts. (See also fig. 8 caption.)

A typical pulse-height spectrum is shown in fig. 8, in which the predicted peak positions of the inelastic alpha groups are indicated.

The cross section for scattering to the 2.31-MeV level is clearly small, but the inability to give a more meaningful upper limit to its value is especially disappointing, since the level has isobaric spin 1 and thus cannot be

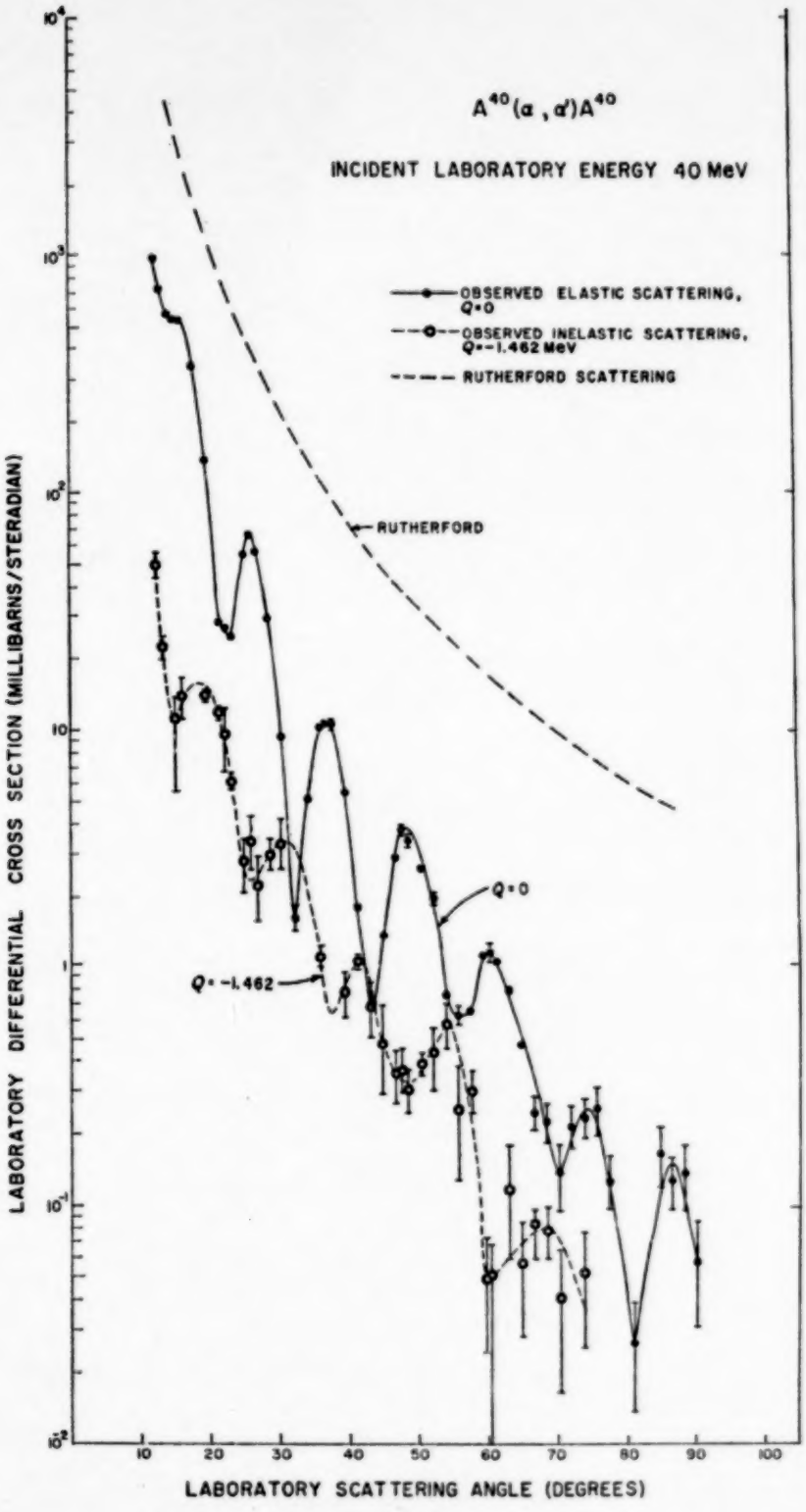

Fig. 11. Laboratory angular distributions for alpha particles scattered elastically and inelastically ($Q = -1.462$ MeV) by A^{40} at incident laboratory energy 40 MeV. The representative errors shown include contributions from counting statistics and subtraction of the effects of neighbouring peaks (fig. 12). The probable error in the scale of absolute cross section is ± 20 %.

excited by nuclear interaction with an alpha particle without a violation of isobaric spin conservation. In previous investigations at 19 MeV [33]) and 31.5 MeV [29]), excitation of this level by alpha particles has also failed to appear.

Angular distributions for elastic and inelastic scattering by O^{16} are

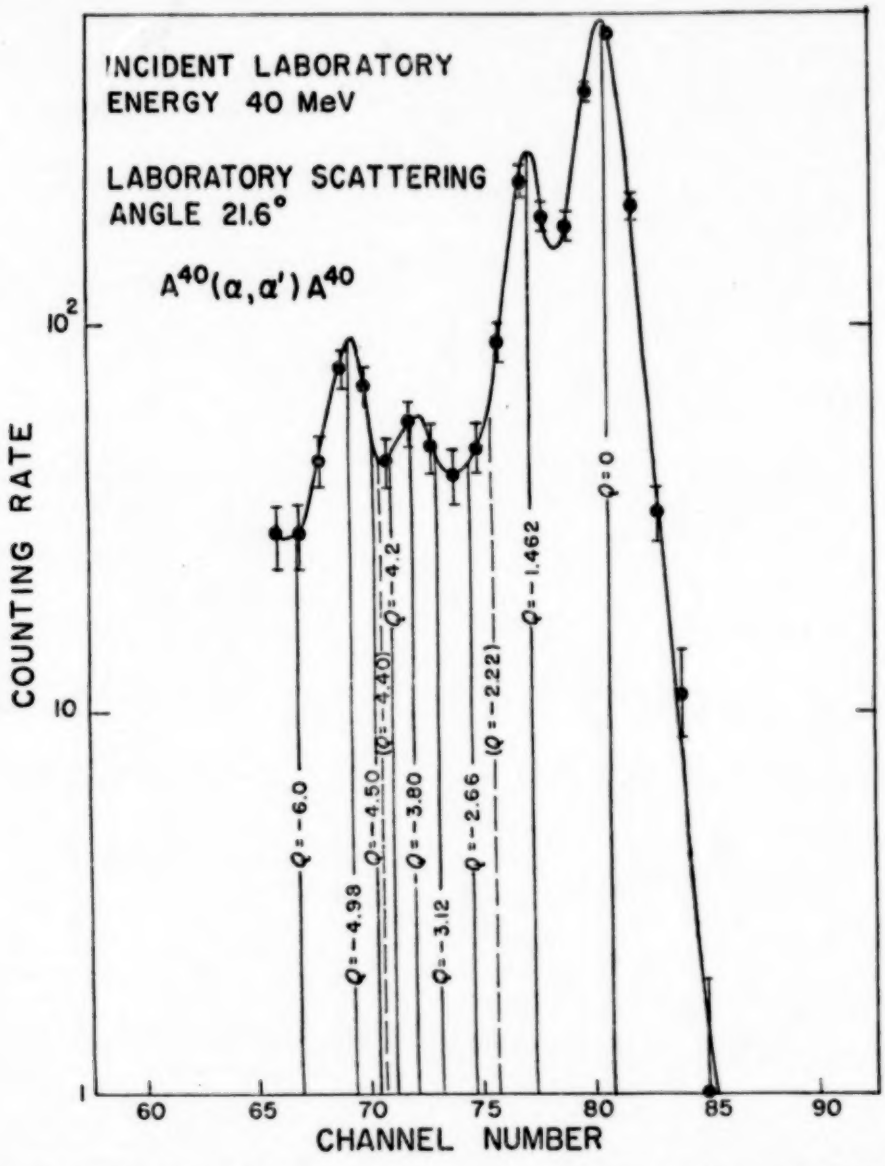

Fig. 12. Pulse-height spectrum for $A^{40}(\alpha, \alpha')A^{40}$ at laboratory scattering angle 21.6° and incident laboratory energy 40 MeV. The counting rate is expressed in counts per 10^5 monitor counts. (See also fig. 8 caption.)

presented in fig. 9. A typical pulse-height spectrum is shown in fig. 10. Since it was impossible to resolve the two levels of the doublet $Q = -6.06$, -6.14 MeV or those of the doublet $Q = -6.91$, -7.12 MeV, the angular distributions of fig. 9 are plotted as if each doublet were a single level. The difficulty of separating the doublets themselves (fig. 10) gives rise to the

relatively large errors shown in the angular distributions for inelastic scattering.

Fig. 11 shows the observed angular distributions for elastic and inelastic scattering by A^{40}. A typical pulse-height spectrum is given in fig. 12. Only the first excited state group ($Q = -1.462$ MeV) could be resolved clearly over an appreciable range of angles.

4. Discussion

4.1. ELASTIC SCATTERING

General Characteristics

The outstanding feature of all of the angular distributions is the sharply defined diffraction-like structure at the forward angles, with angular intervals approximately constant for each distribution (see below). Carbon, nitrogen and oxygen each show three clearly defined maxima and minima at angles below 60°. Argon shows closer spacing (six complete oscillations between 20° and 90°), as might be expected in view of the larger radius and higher centre-of-mass momentum (or wave number) in this case. Cross sections at angles below the first minimum rise very steeply in the forward direction in all cases. The cross section at the first maximum increases quite regularly from carbon to nitrogen to oxygen to argon, though the ratio to the Rutherford cross section decreases.

All curves have in common a deep minimum in the region 75°—80° (laboratory). (The approximate c.m. angles for C, N, O and A are 98°, 95°, 90° and 87°, respectively.) The dip in the oxygen curve is quite spectacular (< 0.02 mb/sr at the minimum), while that for nitrogen is neither as sharp nor as deep (≈ 0.6 mb/sr) as the others.

At backward angles the cross sections are of the order of 1 mb/sr, and suggestions of oscillatory structure are seen. Average cross sections for nitrogen and oxygen are several times that for carbon. (Argon was not observed beyond 90°.)

Elastic and inelastic scattering of 18-MeV alpha particles by argon is described in ref. [1]). Pronounced diffraction-like maxima and minima are observed which are approximately equidistant in scattering angle. As might be expected with the smaller wave number at 18 MeV, the spacing is much wider than that observed at 40 MeV. Oscillatory structure is also observed in elastic scattering of 18.4-MeV alpha particles by oxygen; in this case, however, the spacing of the minima is less regular than for argon [34]).

Comparison with Simple Models; Interaction Radii

Diffraction Scattering. The sharp diffraction-like structure of the angular distributions and the increasing body of evidence for a very small

penetration depth for alpha particles in nuclear matter suggests a comparison of the experimental results with the predictions of a simple diffraction calculation. The diffraction pattern for a black sphere is expressed in terms of the first order Bessel function $J_1(x)$, where the argument x is taken variously as $kR\theta$, $kR \sin \theta$, $2kR \sin \frac{1}{2}\theta$, or even $2kR \tan \frac{1}{2}\theta$, depending upon the choice of shadow plane. (Here k is the c.m. wave number, R is the effective interaction radius, and θ is the c.m. angle of scattering.) Since in the present results the diffraction spacing is most nearly constant in $\sin \frac{1}{2}\theta$ (see below), it seems logical to choose $2kR \sin \frac{1}{2}\theta$ as the argument. The differential cross section is then given by

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{4} k^2 R^4 (4 \cos^2 \tfrac{1}{2}\theta) \left[\frac{J_1(2kR \sin \frac{1}{2}\theta)}{2kR \sin \frac{1}{2}\theta}\right]^2. \quad (1)$$

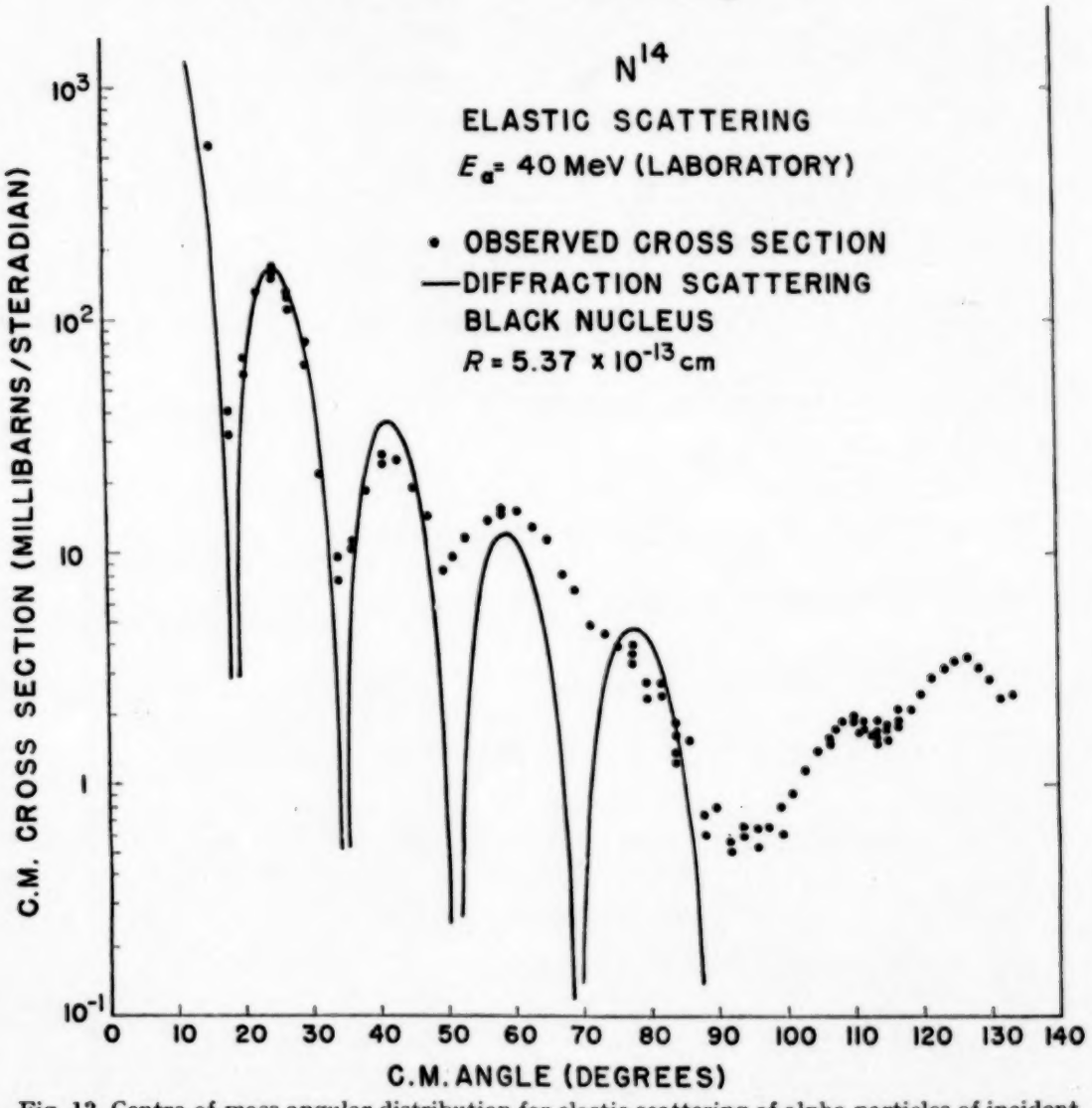

Fig. 13. Centre-of-mass angular distribution for elastic scattering of alpha particles of incident laboratory energy 40 MeV by N^{14}. The solid curve represents diffraction scattering by a black sphere of effective interaction radius 5.37×10^{-13} cm. Absolute cross sections are compared.

(The obliquity factor $4\cos^2\frac{1}{2}\theta$ is appropriate to the choice of the argument $2kR\sin\frac{1}{2}\theta$; this choice applies to integration over the circle of intersection with the black sphere of a plane which is perpendicular to the plane of

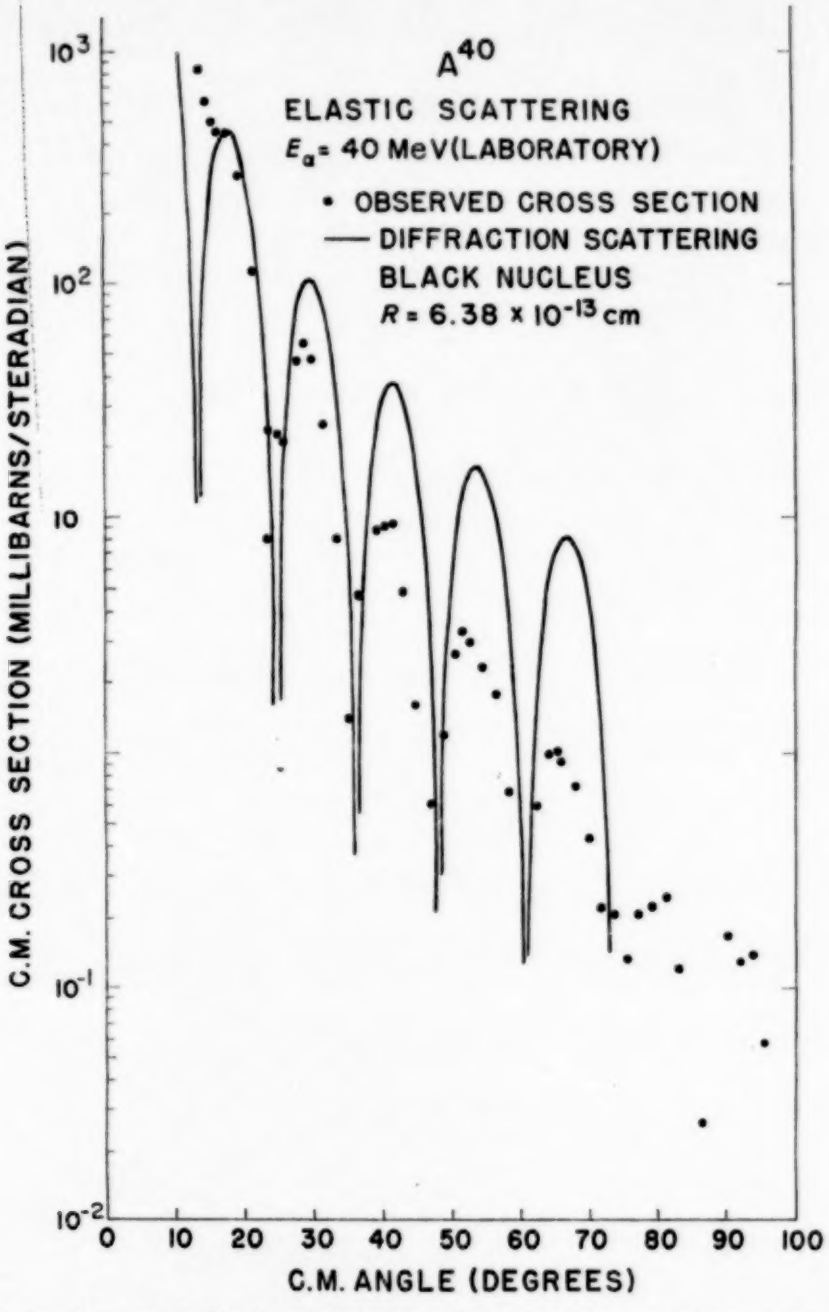

Fig. 14. Centre-of-mass angular distribution for elastic scattering of alpha particles of incident laboratory energy 40 MeV by A^{40}. The solid curve represents diffraction scattering by a black sphere of effective interaction radius 6.38×10^{-13} cm. Absolute cross sections are compared.

incoming and outgoing wave vectors and which contains the momentum transfer vector.)

A comparison of the N^{14} data with the differential cross sections given by

eq. (1) is shown in fig. 13. There is no arbitrary normalization factor: absolute cross sections are compared, and are surprisingly similar. The interaction radius $R = 5.37 \times 10^{-13}$ cm is chosen to reproduce the spacing between successive minima and successive maxima as closely as possible; this choice also gives the locations of the first three minima and maxima to within 1°.

A similar comparion is given for A^{40} in fig. 14. A choice of $R = 6.38 \times 10^{-13}$ cm gives the best spacing; this leads to a small average displacement (+1.3°) in the predicted pattern relative to the observed distribution. The first minimum and maximum are somewhat obscured, presumably by a Coulomb scattering contribution to the cross section which becomes significant at very small angles. The very regular oscillatory structure persists over the entire range of observation, though the average cross section falls by a factor 10^4.

Similar calculations were made for C^{12} and O^{16}. Details of comparions with the observed data will not be given here except in the summary given in table 1, which lists a number of quantities or parameters of interest in the diffraction interpretation of the elastic scattering data.

TABLE 1

Summary of diffraction elastic scattering data

	C^{12}	N^{14}	O^{16}	A^{40}
No. maxima used	3	3	3	5
No. minima used	3	3	3	4
Average $\Delta\theta$	18.6°	16.5°	15.1°	12.1°
% rms deviation in $\Delta\theta$	2.8	4.5	7.3	7.7
% rms deviation in $\Delta \sin \frac{1}{2}\theta$	3.8	3.2	5.4	5.0
Interaction radius $R(10^{-13}$ cm)	[a] 5.11 ± 0.20	5.37 ± 0.17	5.64 ± 0.29	6.38 ± 0.32
$1.35 A^{\frac{1}{3}} + 2.0 (10^{-13}$ cm)	5.09	5.25	5.40	6.62
θ_1(calc.) $-\theta_1$(obs.)	2.6°	0.3°	0.5°	$\approx 1°$
$\delta\theta$	2.2°	0.3°	0.4°	1.3°
$\sigma(\theta_1)$ (calc.) (mb/sr)	118	165	214	458
$\sigma(\theta_1)$ (obs.) (mb/sr)	138	166	175	445
$f(\theta_1)$	1.17	1.01	0.82	0.97
f(av.)	1.05	1.00	0.72	0.41

[a] See text for discussion of method of determination of this radius.

θ = c.m. scattering angle.

$\Delta\theta$ = increment in θ for an interval between minima or between maxima.

$\Delta \sin \frac{1}{2}\theta$ = increment in $\sin \frac{1}{2}\theta$ for an interval between minima or between maxima.

θ_1 = c.m. scattering angle at first maximum; $\sigma(\theta_1)$ = differential cross section at θ_1.

$\delta\theta$ is average value of θ (calc.) $-\theta$(obs.) for all maxima and minima used in determining $\Delta(\sin \frac{1}{2}\theta)$, i.e., the average shift of one pattern relative to the other.

$f(\theta_1)$ is the ratio σ(obs.)/σ (calc.) at first maximum. f(av.) is the ratio averaged over all maxima used.

The number of maxima and minima used to calculate the average interval in $\sin\frac{1}{2}\theta$, and hence ultimately to determine the interaction radius R, is shown in table 1 for each case. The regularity of the spacing in θ is indicated by the rms deviation (%) for $\Delta\theta$; in $\sin\frac{1}{2}\theta$, by the rms deviation (%) for $\Delta \sin\frac{1}{2}\theta$.

C^{12} is the only case for which greater regularity is shown in θ than in $\sin\frac{1}{2}\theta$. We note also that the difference between predicted and observed positions of maxima and minima is largest ($\delta\theta = 2.2°$, table 1) for this case. For these reasons the interaction radius listed for C^{12} is the average of two values: 4.94×10^{-13} cm, which gives the best agreement with the spacing of successive maxima and minima, and 5.27×10^{-13} cm, which gives the best locations for first and second maxima and minima.

The errors listed for the interaction radii correspond to the rms deviations in $\Delta\sin\frac{1}{2}\theta$, though statistical justification could be made for reducing them because several measurements were made in each case. However, no error indicated by this choice exceeds ±5 %.

Fitting of the observed interaction radii with the usual formula of the form $R = r_0A^{\frac{1}{3}}+b$ does not seem very meaningful for such low values of A where gross changes in structure may occur for small changes in A. Values of R are, on the average, about 0.7×10^{-13} cm larger than those given by the optical model parameters [4]) $R= (1.35\ A^{\frac{1}{3}}+1.3)\times10^{-13}$ cm; the latter are based principally on medium and heavy elements, however, and would be expected in any case to be smaller than radii observed here because they represent "average" radii for the nuclear potentials. If a larger intercept ($b = 2.0$) is taken with the same slope ($r_0 = 1.35$), the present radii all fall within ±5 % of the predicted values. A least squares analysis of the present data, including also S^{32}, [18]) gives $R = (1.017\ A^{\frac{1}{3}}+2.94)\times10^{-13}$ cm. The largest deviations from this average curve are for $O^{16}(+2.5\ \%)$ and $C^{12}(-1.8\ \%)$; the rms deviation for all observed radii is 1.5 %.

In contrast to recent results [30]) at 19 MeV, the present results give a smaller interaction radius for C^{12} than for O^{16}.

The average ratio of observed to calculated differential cross section decreases rather regularly [f(av.), table 1] from C^{12} to A^{40}. The ratio at the first maximum, $f(\theta_1)$, shows a similar trend, with a reversal at A^{40} due to the strong forward rise already noted. A^{40} exhibits the most rapid decrease in cross section (relative to the diffraction cross section) with increasing angle.

The sharp and regular diffraction-like structure for all four angular distributions, together with the near equivalence in each case of the observed cross sections to the calculated diffraction cross sections in the neighbourhood of the maxima at small angles, might be interpreted as additional evidence for the essential blackness of nuclear matter for alpha particles at intermediate energies.

Analysis of the data in terms of optical model parameters has not been made.

Born Approximation. The basic assumption underlying the Born approximation (Ba) for elastic scattering is that the interaction potential responsible for the scattering is small in comparison with the kinetic energy of the colliding particles. This implies that the distortion of the incident wave is very small.

For a square well of depth V_0, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left[\frac{2\mu V_0 R^3}{\hbar^2}\right]^2 \left[\frac{j_1(KR)}{KR}\right]^2, \tag{2}$$

where μ is the reduced mass; R is the radius of interaction; $\hbar K$ is the momentum transferred, i.e., $K = 2k \sin \frac{1}{2}\theta$, where k is the c.m. wave number and θ is the c.m. scattering angle; and j_1 is the first-order spherical Bessel function. We note at once that the angular dependence of eq. (2) is very similar to that of eq. (1). The only difference is the replacement of J_1 by j_1; the arguments of the two functions are equal and the predicted spacing of the minima in the angular distributions is the same for both cases. However, the physical assumptions underlying the two models are different and almost mutually exclusive, since the diffraction model is based upon complete absorption (black nucleus), which is not compatible with the Ba assumption of small distortion of the incident wave.

Eq. (2) is used to measure the interaction potential for Ba by substituting, in the left side, the value of the observed differential cross section at the first or second peak of the angular distribution. As with the diffraction model, a radius of interaction can be found by fitting locations of predicted maxima and minima to those of observed maxima and minima in the elastic scattering angular distributions. The applicability of the Ba for high incident energies ($kR \gg 1$; the range is 10—17 here) is then examined by finding whether the validity criterion

$$V_0 \ll \hbar^2 k/\mu R \tag{3}$$

is satisfied.

If the interaction radius R is determined by a best match of the positions of observed and calculated (Ba) first or second minima or maxima, and the square well depth V_0 is evaluated by use of eq. (2) with the observed cross section at the first (C^{12}, N^{14}, O^{16}) or second (A^{40}) maximum, results are as summarized in table 2.

Three circumstances are noted from these results and from a study of positions of successive minima in the calculated distributions.

First, the interaction radii are, on the average, 13 % larger than those determined from the diffraction calculation and, to a lesser extent, larger than those determined from inelastic scattering data (see below).

TABLE 2

Ba results for elastic scattering

	C^{12}	N^{14}	O^{16}	A^{40}
Interaction radius $R(10^{-13}$ cm)	5.75 ± 0.25	6.07 ± 0.15	6.35 ± 0.15	7.25 ± 0.10
Square well depth V_0 (MeV)	4.7	4.2	3.7	3.3

Second, the validity criterion is violated or very poorly satisfied in every case; for example, $V_0 = 3.3$ MeV for A^{40} is incompatible with the requirement $V_0 \ll 4.0$ MeV of eq. (3). It might also be noted that V_0 is very small compared to the potentials required for optical model analysis [4]).

Third, the average separation of successive maxima and minima is poorly given once R is chosen to give the correct location for a particular maximum. For example, if for A^{40} the interaction radius R is chosen so that observed and calculated second maxima occur at the same angle, the locations of observed and calculated fifth maxima differ by 4°. On the other hand, serious dislocation of the whole pattern results if R is chosen to give the observed separations of successive maxima or minima; for example, if the interaction radius for A^{40} determined by this criterion is used to predict the distribution in terms of j_1, the pattern is shifted about 4° (toward large θ) from the observed pattern. (This discrepancy can be easily understood in terms of the asymptotic phase difference of $\frac{1}{4}\pi$ between the Bessel function J_1 and the spherical Bessel function j_1.)

For these reasons we conclude that although the Ba interpretation may apparently be applicable because a reasonable though rather large interaction radius can be deduced from the spacing of minima and maxima in the observed angular distributions, it is in fact not a justifiable approach to the interpretation of elastic scattering of intermediate energy alpha particles by light nuclei.

4.2. INELASTIC SCATTERING

General Remarks

Examination of the inelastic scattering data for C^{12} (figs. 4 and 5), O^{16} (fig. 9) and A^{40} (fig. 11) reveals several features which have sometimes been found in other investigations of inelastic scattering of alpha particles. First, most angular distributions are strongly peaked in the forward direction, and the cross sections are in many cases large (> 10 mb/sr at angles less than about 20°). Second, oscillatory structure is observed in most cases. The oscillations are in several instances almost exactly "out of phase" with oscillations of the corresponding elastic angular distribution (e.g., A^{40},

$Q = -1.462$ MeV; C^{12}, $Q = -7.65$ MeV). Third, cross sections are often surprisingly high at quite large angles; for example, the C^{12} inelastic cross section ($Q = -4.43$ MeV) has an average value of more than 3 mb/sr between 90° and 130° (laboratory), while the elastic cross section is a full order of magnitude less. Fourth, in the few cases for which data beyond 90° (c.m.) are available, no instance of the symmetry about 90° required by a compound nucleus model is observed, though a small compound nucleus contribution cannot be ruled out.

Analysis of Experimental Results

We have attempted to fit the observed angular distributions to the predictions of direct interaction models in order to establish or to verify spin and parity assignments, to estimate interaction radii, and if possible to draw conclusions about the nature of the specific nuclear excitations and inelastic scattering interactions concerned.

If the surface interaction model of Austern, Butler and McManus [8]) (ABM) is applied to inelastic scattering of alpha particles by nuclei with ground state spin zero (C^{12}, O^{16} and A^{40} in the present experiments), the predicted differential cross section takes the form

$$\frac{d\sigma}{d\Omega} = C j_I^2(KR), \tag{4}$$

where C is a constant; I is the spin of the state in which the residual nucleus is left; j_I is the Ith order spherical Bessel function; $\hbar K$ is the momentum transferred; and R is the distance from the centre of the target nucleus at which the (localized) interaction between the incident alpha and a target nucleon takes place.

A similar result is obtained from application of the alpha particle model of the nucleus to the inelastic scattering process [9]), though the physical basis for the calculation is quite different; here it is assumed that the interaction is between the incident alpha particles and one of the alphas of the struck "poly-alpha molecule." Such a model might especially be expected to have possible merit in cases such as C^{12} (three alphas, in this model) and O^{16} (four alphas). In such cases ($Z = N =$ multiple of 2), the predicted differential cross section for rotational excitation is

$$\frac{d\sigma}{d\Omega} = C' \frac{\text{"}d\sigma_{\alpha\alpha}\text{"}}{d\Omega} j_I^2(Kd_\alpha), \tag{5}$$

where C' is a constant; "$d\sigma_{\alpha\alpha}$"/$d\Omega$ is an effective intranuclear differential cross section for alpha-alpha scattering; I is the spin of the state in which the residual nucleus is left; j_I is the $I$the order spherical Bessel function; $\hbar K$ is the momentum transferred in the scattering process; and d_α is the separation

between two neighbouring alphas in the nucleus. If "$d\sigma_{\alpha\alpha}$"/$d\Omega$ is assumed to vary slowly with energy and angle, the angular distributions predicted by eq. (5) are similar to those predicted by eq. (4); however, the alpha particle model has the advantage of lending theoretical palatability to the relatively high observed cross sections for large-angle (high momentum transfer) inelastic scattering. Other collective models share this advantage and give similar predictions for the inelastic alpha angular distributions for nuclei investigated in the present experiments.

Figs. 15—18 show the observed centre-of-mass angular distributions of alpha particles inelastically scattered from C^{12}, O^{16} and A^{40}. Each figure also shows arbitrarily normalized angular distributions calculated according to eq. (4). [We shall henceforth refer to this equation as the ABM result, although the particular type of surface interaction assumed by Austern *et al.* is not the only one which leads to the angular dependence given by eq. (4).] For I, the spin of the final state of the residual nucleus was used; R was taken to be that radius of interaction which would best fit the observed locations of as many small-angle maxima and minima as possible. Where the observed angular distribution could be fitted by curves for several values of I (each with an appropriate value of R), all curves were plotted; those which required a radius R which differed by more than 1×10^{-13} cm from the interaction radius for elastic scattering were discarded. When two levels were too close together to enable resolution of the alpha particles scattered from each one alone (as in the case of O^{16}), the observed angular distributions were plotted for the doublet as if it were a single level and comparisons were made with $j^2_{I_1}(KR_1)$ and $j^2_{I_2}(KR_2)$. (Here I_1 and I_2 are the known spins of the two levels and R_1 and R_2 are the corresponding radii for best fit.) Whenever it was found possible to fit the observed low-angle peaks and valleys of a doublet angular distribution with only one of two possible Bessel functions, it served as an indication that the "fractional excitation" of one of the levels is higher than that of the other.

The fractional excitation is a quantity which serves as a measure of the probability for an inelastic process to go through a specific reaction channel. The differential cross sections for the channel under observation are integrated over the observed range of scattering angle. This integration is then extrapolated towards 0° and 180°, using for the differential cross section the average value over the angular range covered by the experiment. [The integrand contains the factor $\sin\theta$ in the expression for the element of solid angle; this greatly reduces the importance of the unknown extreme forward and backward contributions to the integrated cross sections.] The ratio of this integrated cross section (from 0° to 180°) to πR^2 is the fractional excitation; R is taken to be the average interaction radius for all the inelastic processes observed in the present experiment for a given target

nucleus; in the limit of total absorption of the alpha particles impinging on the nucleus, πR^2 is the sum of the integrated cross sections for all the channels into which subsequent decay can occur.

C^{12}

The analysis of the observed angular distribution of alpha particles scattered inelastically from C^{12}, leaving the residual C^{12} nucleus in the (2^+) 4.43-MeV excited state, was done for incident laboratory energy 41 MeV (fig. 15). The radius of interaction was found from the ABM result

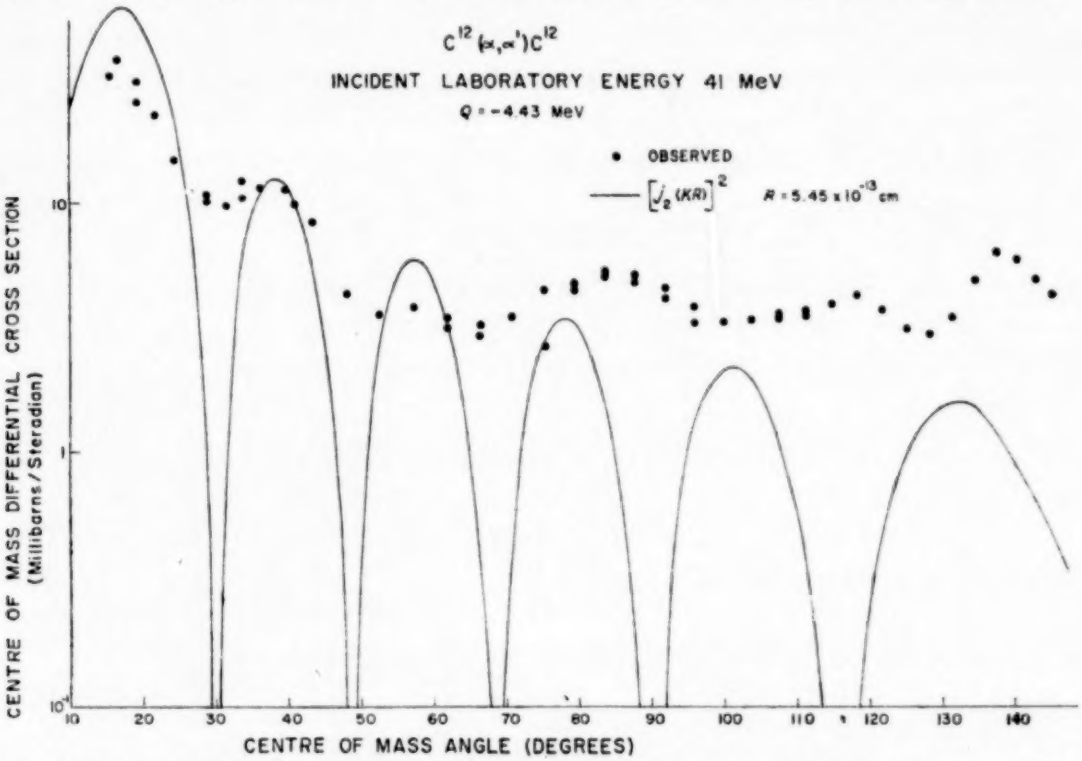

Fig. 15. Centre-of-mass angular distribution of alpha particles inelastically scattered from C^{12}, leaving the residual C^{12} nucleus in the (2^+)4.43-MeV excited state. The incident laboratory energy of the alphas is 41 MeV. The single points are the observed differential cross sections. The solid curve is the square of the second-order spherical Bessel function of the argument KR, where $\hbar K$ is the momentum transferred and R is the interaction radius (here taken to be 5.45×10^{-13} cm). Normalization of the calculated curve to the experimental points is arbitrary.

$d\sigma/d\Omega \propto j_2^2(KR)$, from locations of the first two maxima and first two minima, to be $(5.45\pm0.15)\times10^{-13}$ cm. The arbitrarily normalized ABM result for this interaction radius is shown in fig. 15 as the solid curve; there is little similarity between the ABM curve and the experimental data beyond 50°. It is impossible to explain the general leveling off of the angular distribution beyond 50° and the slight rise beyond 130° on the basis of the ABM analysis. The measured fractional excitation is 6.9 ± 1.0 percent.

The analysis of the observed angular distribution for the residual level at 9.61 MeV was done for incident laboratory energy 40 MeV (fig. 16). Ajzenberg and Lauritsen [22])do not list the angular momentum and parity of

this level. Several suggestions for the spin and parity have been made. Fregeau [35]) suggests 0^+ or 2^+, whereas Vaughn [28]) and Strauch and Titus [36]) suggest 1^-. Even 4^+ is mentioned. It is impossible to fit any peak or valley of $j_1^2(KR)$ to peaks and valleys of the observed angular distribution using a radius of interaction within 1×10^{-13} cm of the radius 5.11×10^{-13} cm for best fit for the elastic scattering (table 1). The first two minima give a

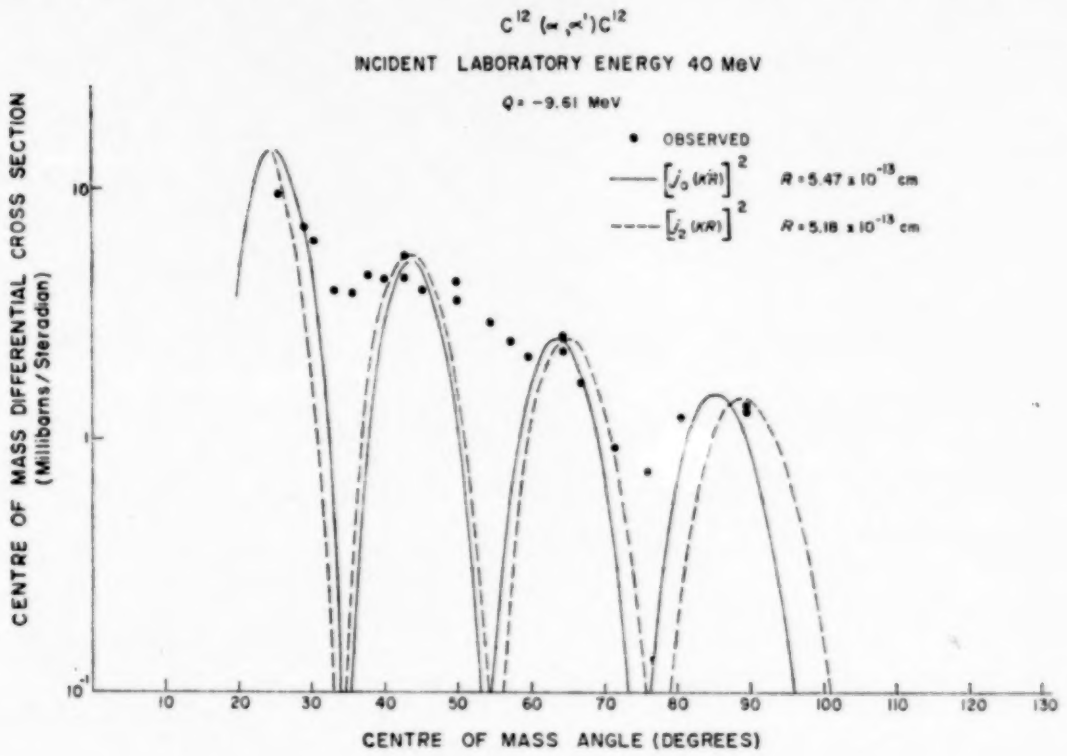

Fig. 16. Centre-of-mass angular distribution of alpha particles inelastically scattered from C^{12}, leaving the residual C^{12} nucleus in the 9.61-MeV excited state. The incident laboratory energy of the alphas is 40 MeV. Normalization of the calculated curves to the observed differential cross sections is arbitrary.

radius of $(5.47\pm0.15)\times10^{-13}$ cm when $j_0^2(KR)$ is used and $(5.18\pm0.15)\times10^{-13}$ cm when j_2^2 is used. The arbitrarily normalized curves for these two choices are shown in fig. 16. It is impossible to fit the observed angular distribution with $j_4^2(KR)$. We conclude that the assignments 0^+ and 2^+ are compatible with the present results in the direct interaction model interpretation, but that 1^- and 4^+ are not. The measured fractional excitation is 4.9 ± 0.7 percent.

No attempt is made here to analyze the observed angular distribution for the residual level at 7.65 MeV, since a more extensive investigation has been made in this laboratory by Bodansky and Eccles [26]). The observed angular distribution agrees satisfactorily with their results, which indicate a 0^+ or 2^+ assignment in the ABM analysis.

O^{16}

Fig. 17 shows the observed centre-of-mass angular distribution of alpha particles inelastically scattered from O^{16}, leaving the residual O^{16} nucleus in either the (0^+) 6.06-MeV or the (3^-) 6.14-MeV excited state. When the fractional excitation for the 6.14-MeV level was assumed to be zero, the radius of interaction was found from the ABM result $j_0^2(KR)$, from locations of the first three peaks and three valleys, to be $(5.36\pm0.20)\times10^{-13}$ cm.

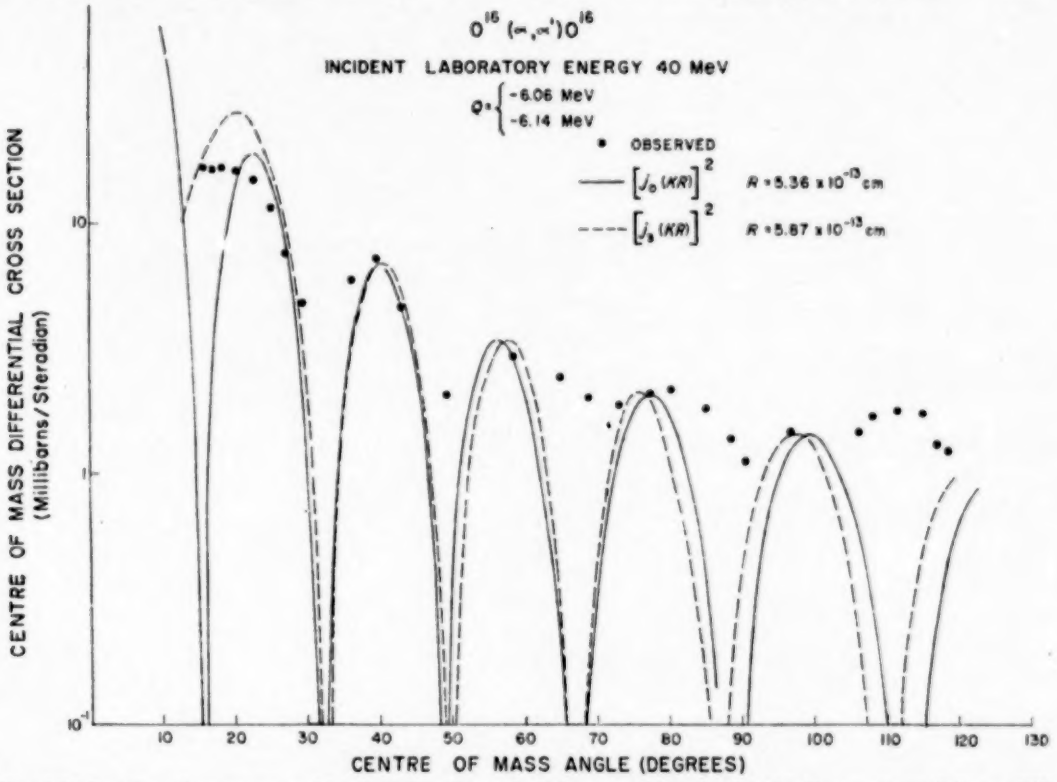

Fig. 17. Centre-of-mass angular distribution of alpha particles inelastically scattered from O^{16}, leaving the residual O^{16} nucleus in the 6.06-MeV and 6.14-MeV excited states. The incident laboratory energy of the alphas is 40 MeV. Normalization of the calculated curves to the observed differential cross sections is arbitrary.

When the fractional excitation for the 6.06-MeV level was assumed to be zero, the radius of interaction was found from the ABM result $j_3^2(KR)$ to be $(5.87\pm0.20)\times10^{-13}$ cm. There is little similarity between the two ABM curves and the experimental data beyond 70°. On the basis of observed data, it is impossible to determine the relative values of the fractional excitations and thus obtain a more meaningful radius of interaction; however, the two values found are compatible with $R = 5.64\times10^{-13}$ cm for elastic scattering. The sum of the fractional excitations for the doublet is 4.4 ± 0.7 percent.

Although evidence of oscillatory structure is seen in the angular distribution (fig. 9) for scattering to the two levels at 6.91 MeV (2^+) and 7.12 MeV (1^-), a good spherical Bessel function fit to the observed data cannot be made. However, there is some indication that a $j_1^2(KR)$ curve with $R =$

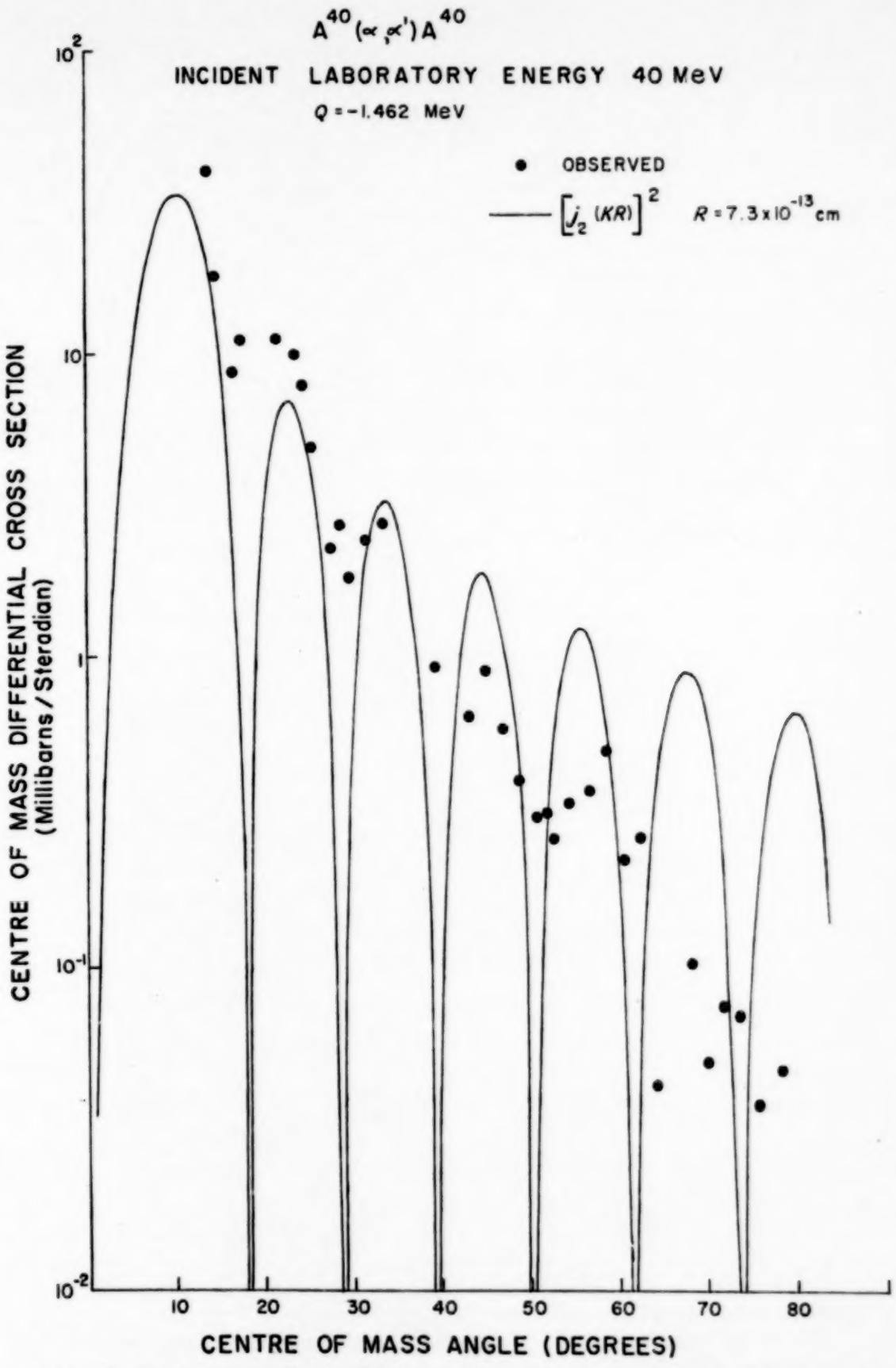

Fig. 18. Centre-of-mass angular distribution of alpha particles inelastically scattered from A^{40}, leaving the residual A^{40} in the 1.462-MeV excited state. The incident laboratory energy of the alphas is 40 MeV. Normalization of the calculated curve to the observed differential cross sections is arbitrary.

6.0×10^{-13} cm is a possible though poor choice. Since no possible choice was found for a transition to the even parity level at 6.91 MeV, it is probable that most of the fractional excitation of 3.5 ± 0.5 percent for the doublet is due to excitation of the 7.12-MeV level.

A^{40}

Fig. 18 shows the observed centre-of-mass angular distribution for alpha particles inelastically scattered from A^{40}, leaving the residual nucleus in the (2^+)1.462-MeV excited state. The interaction radius was found from the ABM result $j_2^2(KR)$, from the locations of the first four valleys and three peaks, to be $(7.3\pm0.2)\times10^{-13}$ cm, which is about 0.9×10^{-13} cm larger than that found for elastic scattering. This difference is the largest one observed in the present experiments. The rapid decrease of the differential cross sections with the increase in scattering angle is not incompatible with the single-nucleon excitation of the ABM model; however, the elastic angular distribution falls equally fast, and collective excitations may be equally probable. Observations at larger angles are needed. The measured fractional excitation is 1.4 ± 0.4 percent.

5. Summary and Conclusions

A summary of the results of the analyses of the experimental angular distributions is given in table 3, where characteristics of levels and values of measured quantities are listed.

Although each method of establishing an interaction radius determines the radius to within about ±4 percent or less, it is evident from the data of table 3 that ABM analysis of inelastic scattering data yields interaction radii consistently higher than those found from diffraction analysis of elastic scattering data, and that the discrepancy may be as much as several times the uncertainty in either determination. [The Ba elastic scattering radii are even higher, but we do not consider them meaningful (sec. 4).] The present results are not unusual in this respect [2,17–19]). The discrepancy is largely removed when a diffraction analysis [13]) of the inelastic angular distributions is made.

The high fractional excitations observed for a number of inelastic scattering reactions (table 3), together with the sharply defined oscillatory structure of the corresponding angular distributions, imply very strong interactions between alphas and nuclear matter, localized near the nuclear surface. The direct interaction model analysis enables spin and parity assignments to be made and gives reasonable fits to the observed angular distributions. However, the incompatibility of the assumptions of short range forces and small distortion of incident and scattered waves; the discrepancy in inter-

action radii noted above; and, finally, the high cross sections often observed for large scattering angles in inelastic scattering of alphas by light elements, suggest that a more realistic interpretation is needed.

A black nucleus diffraction scattering interpretation of the present elastic scattering results gives reasonable magnitudes for cross sections and

TABLE 3

Summary of experimentally determined quantities in elastic and inelastic scattering

Target	Excitation energy of residual nucleus (MeV)	Spin and parity	Refs.	Type of analysis	Interaction radius (10^{-13} cm)	Fractional excitation (percent)
C^{12}	0	0^+	22	diff.	5.11 ± 0.20	
	0	0^+	22	Ba	5.75 ± 0.25	
	4.43	2^+	22	ABM	5.45 ± 0.15	6.9 ± 1.0
	7.65	0^+	26	ABM	5.9 ± 0.2	
		2^+	26	ABM	5.6 ± 0.2	
	9.61	0^+	p	ABM	5.47 ± 0.20	
		2^+	p	ABM	5.18 ± 0.20	4.9 ± 0.7
N^{14}	0	1^+	22	diff.	5.37 ± 0.17	
	0	1^+	22	Ba	6.07 ± 0.15	
O^{16}	0	0^+	22	diff.	5.64 ± 0.29	
	0	0^+	22	Ba	6.35 ± 0.15	
	6.06	0^+	22	ABM	5.36 ± 0.20	4.4 ± 0.7
	6.14	3^-	22	ABM	5.87 ± 0.20	
	6.91	2^+	22	ABM	—	3.5 ± 0.5
	7.12	1^-	22	ABM	6.0 ± 0.2	
A^{40}	0	0^+	23	diff.	6.38 ± 0.32	
	0	0^+	23	Ba	7.25 ± 0.10	
	1.462	2^+	23	ABM	7.3 ± 0.2	1.4 ± 0.4

Targets are listed in column 1. The references for excitation energies (column 2) and spins and parities (column 3) are listed in column 4; the letter p indicates heretofore unassigned spins and parities which are consistent with the present experiments. The types of analysis used are the diffraction model (diff.), the Born approximation (Ba), and the direct interaction model (ABM); they are discussed in sec. 4 and indicated here in column 5. The interaction radius for best fit to the theoretical angular distribution is listed in column 6. The observed fractional excitation for each inelastic scattering reaction (the ratio of the integrated cross section for the particular inelastic reaction to πR^2) is given in column 7.

accurate locations of maxima and minima in the angular distributions over a surprisingly large range of angles. The Born approximation treatment for elastic scattering is found to be unsatisfactory.

The symmetry about 90° required by a compound nucleus model for elastic or inelastic scattering is not observed in any of the reactions studied, although a small contribution from compound nucleus processes is in no instance excluded.

It is a pleasure to acknowledge the generous assistance of Professors J. S. Blair, E. M. Henley and B. A. Jacobsohn, given in several discussions of interpretation of the experimental results. Thanks are also due to Dr. M. E. Rickey, Mr. Robert Cole, Mr. Ira Karp, Professor T. J. Morgan, and to the technical staff of the University of Washington Cyclotron for their willing and competent collaboration in various phases of the experimental work.

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CONNECTION BETWEEN MASSES AND ISOBARIC SPIN

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Abstract: A fundamental theory of the elementary particles is presented assuming a dependence of the function describing a particle on two space-time variables and symmetry of the fundamental equations between them. All masses, with the exception of charged pions and muons, but including the electron mass of purely electro-magnetic origin, are predicted with a remarkable accuracy introducing a universal length as the only adjustable parameter. The strangeness and the Gell-Mann-Nishijima scheme find a simple interpretation; the isobaric spins of the different masses arise as an automatic consequence of the formalism. It is shown that the theory implies an indefinite metric. The existence of a pion with zero isobaric spin is predicted by the theory. In the appendix an attempt is presented to explain the existence of the pions with isobaric spin $T = 1$ and of the muons.

1. Introduction

In this work a fundamental theory of the elementary particles is presented to extend and develop the results obtained in various preceding notes. In sections 2 to 5 the theory previously developed is summarized with particular reference to the postulates and its essential consequences. This exposition is necessary for the sake of completeness and to stress the essential logical unity of the subsequent developments. In section 6 the isobaric spin of an elementary particle is defined, the connection between masses and their isobaric spin is found and the Gell-Mann-Nishijima scheme is obtained. In section 7 it is shown that new selection rules are implied in the theory, which exclude as possible physical products of a π+N reaction the non-observed masses calculated in previous work. Thus, if only pions and nucleons are initially present, no other particles than those already observed can be physically detected. In section 7 it is also shown that an indefinite metric is the necessary framework for the present theory. Finally, in the appendix, an attempt is presented at the explanation of the masses and of other properties of the charged pions and of the muons, which are the only particles not predicted by the preceding theory.

2. The Fundamental Equations

The groundwork of our theory, as was already suggested by some authors †, consists in assuming a function $\psi(x, \eta)$ describing an elementary particle which depends on two space-time variables, an external variable $x_\nu (\nu = 1, 2, 3, 4)$ of the pseudoeuclidean observable space and an internal variable $\eta_\nu (\nu = 1, 2, 3, 4)$ of an internal pseudoeuclidean space connected with the structure of the particle. We shall see in section 6 that the internal space will enable an explanation of the existence of the isobaric spin and this will provide the deep physical meaning of the formalism. However, the isobaric spin will be defined only in section 6 and we proceed here in a somewhat formal way.

The first problem is to find the fundamental equation of our "bilocal" $\psi(x, \eta)$. The simplest hypothesis one can make consists in assuming symmetry between the x and η variables. Since the mass must be an automatic consequence of the theory, the mass term must not appear in the fundamental equation. Then we have in the case of a spin-$\frac{1}{2}$ particle

$$\gamma_\nu \left(\frac{\partial}{\partial x_\nu} + \frac{\partial}{\partial \eta_\nu} \right) \psi(x, \eta) = 0, \tag{1}$$

where ψ has the transformation properties of a spinor, and in the case of a spin 0 particle

$$(\Box_x + \Box_\eta) \psi(x, \eta) = 0, \tag{2}$$

where ψ is a scalar or a pseudoscalar. Naturally symmetry is destroyed when supplementary conditions on the solution ψ are introduced in the theory, in accordance with the different physical meanings of the x and η pseudoeuclidean spaces. Then symmetry concerns essentially the structure of the two spaces in the sense that if a transformation of the Lorentz group is performed in one space, the same transformation must be performed in the other.

† The first idea of introducing a second space-time variable seems due to Markov (ref. [1a])) and other russian physicists (see ref. [1b])). This idea was reexamined by Yukawa (ref. [1c])) and Shirokov (ref. [1d])). The connection between non-local theory and the mass quantization problem was also suggested by Wataghin (ref. [1e])) and Yukawa (ref. [1f])). A generalization of the Dirac equation to the case of an internal and an external variable was proposed by Rayski (ref. [1g])) and subsequent works in Nuovo Cimento and Acta Phys. Polon. doubling the Dirac matrices and connecting the internal space with isobaric spin; he interpreted in this way a theory of Pais (ref. [1h])).

The internal equation of this paper, obtained assuming symmetry between internal and external coordinates and without doubling Dirac matrices, was proposed by the author in ref. [2a]). The interaction with the electromagnetic field was introduced (see ref. [2b])) and the existence of a purely electromagnetic mass derived (see ref. [2c])). The complete calculation of the electromagnetic mass assuming the internal function of a Majorana neutrino as zero approximation was performed (see ref. [2d])). Finally some of the mass selection rules of the present paper were proposed (see refs. [2e, 2f])).

In this paper we are also concerned with the interaction between the electromagnetic field and one particle with spin $\frac{1}{2}$. Let us introduce the operator $A_\nu(x')$ of the quantized electromagnetic field

$$A_\nu(x') = \exp(iH_0t')A_\nu(\mathbf{x}')\exp(-iH_0t'), \tag{3}$$

$$\Box_{x'}A_\nu(x') = 0, \tag{4}$$

where H_0 is the hamiltonian operator of the electromagnetic field in the vacuum. Our symmetry hypothesis leads immediately to the following equation:

$$\gamma_\nu\left(\frac{\partial}{\partial x_\nu} - ieA_\nu(x+\eta) + \frac{\partial}{\partial \eta_\nu}\right)\psi(x,\eta,q) = 0, \qquad (x' = x+\eta) \tag{5}$$

where q indicates the coordinates of the electromagnetic field.

The set of fundamental equations of the present theory will be complete by adding to the preceding equations the supplementary condition

$$\frac{\partial^2}{\partial x_\nu \partial \eta_\nu}\psi(x,\eta,q) = 0, \tag{6}$$

which, according to our symmetry hypothesis, is a generalization of the condition $\eta_\nu p_\nu^{(x)} = 0$ introduced by Yukawa in 1950. Condition (6) is considered as one for limiting the region of variability of the variable η and for defining the internal space. Considering in the rest system of the particle (defined by $\nabla_x\psi = 0$) the intersection with an hyperplane $\eta_4 = i\eta_0 = iA$, where A is an a priori arbitrary real constant, which will be determined by a further condition in section 5, the variability region of the argument η of ψ reduces to a three dimensional region in a four-dimensional space and one can put in the rest system $\partial\psi/\partial\eta_4 = 0$; then condition (6) is satisfied.

As was shown in previous work (ref. [2a]), eq. (8) and ref. [2b]), eqs. (7) and (8)) eqs. (1) and (5) can be separated in the rest system of the particle with respect to the x and η variables, and we assume that only the separated solution, having the form $\psi = \exp(ipx)\,\varphi(\eta)$, has a physical meaning. We anticipate here that eq. (5) with condition (6) enables us to solve the problem of the internal structure of the particle and to calculate its mass when it interacts with its own electromagnetic field, but not to describe the observable motion of the particle in an external electromagnetic field. The observable motion will be described (cf. section 4) by an equation similar to (5) to which the electromagnetic mass term calculated with (5) is added and which must be solved without separating the x and η variables and without imposing condition (6). We shall impose condition (6) exclusively on the solution describing the internal structure of a free particle and the interaction of its own electromagnetic field with its structure.

The distinction between the two problems, the internal problem concerning the mass and the structure of one free particle described by a separated $\psi = \exp(ipx)\varphi(\eta)$ which satisfies the subsidiary condition (6), and the problem of the observable motion described by a non-separated ψ which does not satisfy eq. (6), is an essential feature of the theory and must always be kept in mind.

We can now go on with the solution of equation (1). One can separate this equation in the rest system by putting $\psi = f(x_4)\varphi(\eta)$ and the following internal equation is obtained for $\varphi(\eta)$ (cf. ref.[2a]) eq. (12), ref.[2b]), eq. (7)):

$$-i\alpha \cdot \nabla_\eta \varphi(\eta) = m\varphi(\eta), \tag{7}$$

where the separation constant m is the mass. In the case of a spin 0 particle the internal equation is (cf. ref. [2a]), eq. (6))

$$\Delta_\eta \varphi(\eta) = -m^2 \varphi(\eta). \tag{8}$$

It must be noted that eq. (7) differs from the Dirac equation by the absence of the time derivative term and of the matrix β in the mass term. The latter feature will be of interest in section 6 when the transformation properties of φ will be discussed. Eq. (8) differs from the Klein-Gordon equation by the absence of the derivative term and the sign of the mass term.

We have solved (cf. ref. [2a, b])) the internal equations (7) and (8) with the boundary condition that the internal function $\varphi(\eta)$ vanishes at the outside of a spatial sphere of radius l, defined in the rest system of the particle, where l is a universal length. The above condition has the meaning of a fundamental law of nature: if one assumes the continuity of the internal function on the surface of the sphere an eigenvalue problem for the mass is set up starting from eq. (7) or (8) and the masses of the baryons and of the mesons are obtained; if one assumes the discontinuity on the surface of the sphere of the internal function describing a particle with spin $\frac{1}{2}$, the leptons will be obtained.

3. Baryons and Mesons

Let us first discuss the case of baryons and of mesons. In both cases, with an elementary procedure very similar to the case of the hydrogen atom (cf. ref. [2a, b])) the internal function can be separated in a radial and an angular part. The radial part is determined by a spherical Bessel equation

$$\frac{\mathrm{d}^2 Y_k}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}Y_k}{\mathrm{d}r} + \left(m^2 - \frac{k(k+1)}{r^2}\right) Y_k = 0, \tag{9}$$

and the mass quantization condition is

$$Y_k(ml) = \frac{1}{\sqrt{ml}} J_{k+\frac{1}{2}}(ml) = 0. \tag{10}$$

In eq. (9) k is zero or a positive integer for bosons and a positive or negative integer for fermions; it is connected with the total angular momentum by the relation $k = j$ in the case of bosons and $k = j+\frac{1}{2}$ for $k > 0$ or $k = -j-\frac{1}{2}$ for $k < 0$ in the case of fermions; $j(j+1)$ is the eigenvalue of the square of the total internal angular momentum operator equal to $\mathbf{M} = -i\boldsymbol{\eta}\wedge\nabla_\eta+\frac{1}{2}\boldsymbol{\sigma}$ in the case of fermions or to $\mathbf{M} = -i\boldsymbol{\eta}\wedge\nabla_\eta$ in the case of bosons. Only the spinor part of this angular momentum, if it exists, is observed in the local limit. Indeed, if in eq. (1) a spherically symmetrical interaction in the x and η variables is introduced, $-i\mathbf{x}\wedge\nabla_x-i\boldsymbol{\eta}\wedge\nabla_\eta+\frac{1}{2}\boldsymbol{\sigma}$ or $-i\mathbf{x}\wedge\nabla_x-i\boldsymbol{\eta}\wedge\nabla_\eta$ is a constant of motion for fermions or bosons respectively. In the local limit, where the internal coordinates are neglected, we are left with an angular momentum $-i\mathbf{x}\wedge\nabla_x+\frac{1}{2}\boldsymbol{\sigma}$ or $-i\mathbf{x}\wedge\nabla_x$ and a particle with spin $\frac{1}{2}$ or spin 0 is obtained.

The mass spectrum obtained with the mass quantization condition (10) is degenerate with respect to the boson or fermion character of a particle, because equation (9) holds for particles with spin $\frac{1}{2}$ and with spin 0. Moreover, a further degeneration occurs in the case of fermions with respect to the auxiliary quantum number k because the same eq. (9) is obtained with $k = A$ or $k = -A-1 (A > 0)$. We eliminate both degenerations by the rule that there must correspond to a boson an even positive k, and to a fermion an odd positive k. This rule implies that the internal function is symmetric with respect to a reflection in the η space [3]). We have no theoretical justification for the preceding rule, and it is hoped that in the further development it may be derived from deeper considerations.

We note that a degeneration of the mass spectrum with respect to the orientation of the internal angular momentum still remains. The introduction of the isobaric spin in section 6 will enable us to restrict this degeneration.

The lowest masses predicted by condition (10) with the value 2.30×10^{-13} cm for the universal length are given in tables 1 and 2 for bosons and fermions respectively. The Bessel function of order $-\frac{1}{2}$ is admitted because it gives a finite mass in any volume containing the origin; lower negative orders give an infinite mass in this volume and must be excluded.

The observed particles are given in boldface type. We have a number of predicted particles infinitely greater than the number of observed particles. However, as we shall see in sections 5 and 7, all unobserved particles will be automatically eliminated by simple selection rules implied in the theory (and essentially connected with the definition of strangeness and its conservation in a π+N reaction) and among all particles of tables 1 and 2 only the observed particles will remain.

TABLE 1

Mesons: positive even k and $k = 0$

Internal total angular momentum	Allowed internal intrinsic parity	k	Internal function	Allowed or excluded masses and corresponding internal intrinsic parities (the allowed masses are given in boldface type)				
0	+	0	$J_{-\frac{1}{2}}$	**265**; $-(-1)^{1}$	795; $-(-1)^{4}$	1 325; $-(-1)^{18}$	1 855; $-(-1)^{25}$ ...	
0	−	0	$J_{+\frac{1}{2}}$	530; $-(-1)^{1}$	1 060; $-(-1)^{4}$	1 590; $-(-1)^{10}$	2 120; $-(-1)^{17}$ ...	
2	+	2	$J_{\frac{3}{2}}$	**972**; $-(-1)^{3}$	1 533; $-(-1)^{8}$	2 080; $-(-1)^{16}$	2 615; $-(-1)^{27}$ ...	
4	−	4	$J_{\frac{7}{2}}$	1 380; $-(-1)^{7}$	1 975; $-(-1)^{15}$	2 537; $-(-1)^{25}$	3 085; $-(-1)^{37}$ ...	
6	+	6	$J_{\frac{11}{2}}$	1 768; $-(-1)^{12}$	2 390; $-(-1)^{23}$	2 975; $-(-1)^{35}$	3 530; $-(-1)^{48}$ ...	
...	...	...	...	...	...	...	...	...

TABLE 2

Baryons: positive odd k

Internal total angular momentum	Allowed internal intrinsic parity	k	Internal function	Allowed or excluded masses and corresponding internal intrinsic parities (the allowed masses are given in boldface type)				
$\frac{1}{2}$	+	1	$J_{\frac{1}{2}}$	758; $-(-1)^{2}$	1 302; $-(-1)^{6}$	**1 839**; $-(-1)^{13}$	2 365; $-(-1)^{22}$ ...	
$\frac{3}{2}$	−	3	$J_{\frac{5}{2}}$	1 180; $-(-1)^{5}$	1 757; $-(-1)^{11}$	**2 310**; $-(-1)^{20}$	2 860; $-(-1)^{32}$ ...	
$\frac{5}{2}$	+	5	$J_{\frac{9}{2}}$	1 578; $-(-1)^{9}$	**2 185**; $-(-1)^{19}$	2 758; $-(-1)^{30}$	3 310; $-(-1)^{44}$ ...	
$\frac{7}{2}$	−	7	$J_{\frac{13}{2}}$	1 965; $-(-1)^{14}$	**2 600**; $-(-1)^{28}$	3 170; $-(-1)^{41}$	3 760; $-(-1)^{53}$ ...	
...	...	...	...	...	...	...	...	...

4. Leptons: the Neutrino and the Electron

We now assume that the solution of the spinorial equation (1) is discontinuous on the surface of the sphere; the simplest solution of this kind is

$$\psi(x, \eta) = N u_p f(\eta_\nu^2) \exp\left[-i x_\nu p_\nu^{(x)}\right] \exp\left[-i \eta_\nu p_\nu^{(\eta)}\right], \tag{11}$$

where u_p is a one-column matrix satisfying the equation

$$\alpha_\nu (p_\nu^{(x)} + p_\nu) u_p = 0 \tag{12}$$

and the $p_\nu^{(\eta)} (\nu = 1, 2, 3, 4)$ satisfy the conditions $p_\nu^{(\eta)2} = m_0^2$, $p_4^{(\eta)} = 0$ for $\mathbf{p}^{(x)} = 0$ and are otherwise arbitrary; $f(\eta_\nu^2)$ is constant and different from zero for $\eta_\nu^2 < l^2$, and zero for $\eta_\nu^2 > l^2$; N is a normalization factor.

The above solution can describe a particle with spin $\frac{1}{2}$ and mass m_0, and in the limit $m_0 \to 0$ describes a neutrino with zero mass. The interest of this solution consists in the fact that assuming it as a zero approximation in the perturbation treatment of eq. (5) which describes the interaction between

the bilocal function ψ and the quantized electromagnetic field, the existence of a purely electromagnetic mass can be predicted and the experimental value of the electron mass can be obtained with remarkable accuracy using the previously introduced numerical value of the universal length (cf. ref.[2c,d])).

This remarkable feature of the theory is mainly a consequence of the fact that two distinct physical processes of emission and absorption of virtual photons arise from the interaction of a bilocal function with the electromagnetic field. A first process is obtained by separating in eq. (5) the external from the internal variables in the rest system of the particle (defined by $\nabla_x \psi = 0$) (cf. ref. [2c]), p. 1129), the rest energy being the separation constant: in this way an internal equation is obtained which describes the interaction of photons with the internal part of the bilocal field only. In general one can expand the internal function $\varphi(\eta, q)$ in terms of the eigenfunctions of the mass operator and of the hamilton operator of the pure radiation field:

$$\varphi(\eta, q) = \sum_n \sum_{n'} a_{n,n'} \varphi_n(\eta) \Psi_{n'}(q). \tag{13}$$

Putting $\eta_4 = 0$, $\partial\varphi/\partial\eta_4 = 0$ (cf. section 2) in the rest system of the interacting particle with mass m, the following matrix elements are obtained (cf. ref. [2c]), eq. (9)):

$$\begin{aligned}
(H'^{(\eta)}_{\mathrm{II/I}})_{\mathrm{em}} &= -e\sqrt{\frac{2n}{k}} \exp(-i\mathbf{k}\cdot\mathbf{x}) \int \varphi^+_{\mathrm{II}} \gamma_\alpha \varphi_{\mathrm{I}} \exp(-i\mathbf{k}\cdot\boldsymbol{\eta}) \mathrm{d}^3\eta, \\
(H'^{(\eta)}_{\mathrm{II/I}})_{\mathrm{abs}} &= -e\sqrt{\frac{2\pi}{k}} \exp(i\mathbf{k}\cdot\mathbf{x}) \int \varphi^+_{\mathrm{II}} \gamma_\alpha \varphi_{\mathrm{I}} \exp(i\mathbf{k}\cdot\boldsymbol{\eta}) \mathrm{d}^3\eta.
\end{aligned} \tag{14}$$

Expanding m in power series of e and neglecting transitions between states of different mass one obtains to the second order in e (cf. ref.[2c]), eq. (12))

$$\mp m'' = \sum_1^4 {}_\alpha \sum_k \frac{H'^{(\eta)}_{0/0k\alpha} H'^{(\eta)}_{0k\alpha/0}}{-k}, \tag{15}$$

where the sign of m'' corresponds to a negative or positive energy respectively. A summation on each possible intermediate state of spin and of the sign of energy is understood.

A finite result is obtained for m'' owing to the cut-off of the high momenta of the photon which is an automatic consequence of eq. (5) †; starting from the zero approximation function $\varphi_0(\eta, q) = Nf(\eta_\nu^2) u_\rho \psi_\nu(q)$ one obtains a

† The value of the cut-off momentum depends on the internal functions which appear in the matrix elements and is of the order of l^{-1}. As is well known [4]), it does not affect the results of quantum electrodynamics at low energies.

perturbed solution of the separated internal equation having the form $Nf(\eta_\nu^2)\Sigma a_{i,n} u_i \Psi_n(q)$ where $f(\eta_\nu^2)$ and N are the same as in eq. (11); the calculated mass is $m'' = \pm 3e^2/5l$.

This result remains unchanged in the limit $m_0 \to 0$ and then the mass m'' can be considered of purely electromagnetic origin. On the other hand the unperturbed mass m_0, when interaction exists, loses its meaning of physical mass and it seems reasonable that the limiting process $m_0 \to 0$ can be performed consistently in the rest system of the particle with mass $m = m_0 + m''$ without contradicting relativity, if the physical mass m remains finite.

We now consider the second virtual process which arises from the interaction of photons with a bilocal function, i.e. the emission or absorption of photons interacting with the external part of the bilocal field only. One adds to eq. (5) a term m'' to take into account the contribution due to structure, and then a new perturbation problem is set up for the calculation of the self-energy, without separating internal and external variables, but the time variable only from the other variables, with the energy as separation constant. The new $\psi(x, \eta, q)$ which satisfies to the equation with the mass term is not required to satisfy the subsidiary condition (6). Indeed, the perturbed solution ψ is generally expressed by an expansion in plane waves $\sum_{pnm} \exp(ipx)\varphi_n(\eta)\Psi_m(q)$ and, in the rest system of the particle, generally one has not $\nabla_x \psi = 0$, while the condition $\partial\Psi/\partial\eta_4 = 0$ holds because the region of variability of the η variable is always restricted to the three-dimensional η space, as it was obtained by solving the internal problem with condition (6).

The matrix elements describing the emission or absorption processes are in the present case (cf. ref. [2b]), eq. (17))

$$(H'_{\mathrm{II/I}})_{\mathrm{em}} = -e\sqrt{\frac{2\pi}{k}}\exp(ik_0 t)\delta(\mathbf{p}_{\mathrm{I}}^{(x)} - \mathbf{p}_{\mathrm{II}}^{(x)} - \mathbf{k})\int \varphi_{\mathrm{II}}^{+}\gamma_\alpha\varphi_{\mathrm{I}}\exp(-ik_\nu\eta_\nu)\mathrm{d}^4\eta,$$

$$(H'_{\mathrm{II/I}})_{\mathrm{abs}} = -e\sqrt{\frac{2\pi}{k}}\exp(-ik_0 t)\delta(\mathbf{p}_{\mathrm{I}}^{(x)} - \mathbf{p}_{\mathrm{II}}^{(x)} + \mathbf{k})\int \varphi_{\mathrm{II}}^{+}\gamma_\alpha\varphi_{\mathrm{I}}\exp(ik_\nu\eta_\nu)\mathrm{d}^4\eta, \tag{16}$$

where conservation of momentum holds between the momentum of the photon and the momentum of the motion of the particle in the observable space. The four-dimensional integration reduces to an integration on a three-dimensional variety in the four-dimensional space when the supplementary condition $\eta_4 = \text{const}$ for $\mathbf{p}^{(x)} = 0$ is taken into account (cf. ref. [2b]), p. 974).

The numerical result for the mass is in complete agreement with experiment if in the calculation for m'' one assumes that the zero approximation function (11) satisfies the Majorana condition, i.e. that the momentum $\mathbf{p}^{(x)} + \mathbf{p}^{(\eta)}$ is parallel or antiparallel to the z-component of the spin; then in

the rest system $p_1^{(x)} = p_1^{(\eta)} = p_2^{(x)} = p_2^{(\eta)} = p_3^{(x)} = 0$, $p_3^{(\eta)} = m_0$ and the arbitrariness of the $p_\nu^{(\eta)}$ is eliminated. Even with the validity of the Majorana condition the matrix element (14) of the electromagnetic interaction $e\boldsymbol{\alpha} \cdot \mathbf{A}$ between the photons and the internal function is not zero, as in the usual theory. Indeed in our case the matrix element corresponding to $\alpha = 3$ is different from zero because the momentum $\mathbf{k}$ of the photon is not necessarily parallel to $\mathbf{p} = \mathbf{p}^{(x)} + \mathbf{p}^{(\eta)}$ (and then the electromagnetic field $\mathbf{A}$ is not perpendicular to $\mathbf{p} = \mathbf{p}^{(x)} + \mathbf{p}^{(\eta)}$) as in the case of the interaction of photons with the external part of $\psi(x, \eta)$, where conservation of momentum holds between the momentum of the photon and the momentum of the observable motion of the particle.

For the details of the calculations, see ref.[20,d]). One obtains $m'' = \pm 4e^2/5l$ i.e. $\frac{4}{3}$ of the mutual electrostatic energy of classical electrodynamics. This result is the same as that of the classical, non-relativistic, theory of the Lorentz electron. The contribution of the external part, obtained by solving the equation with the mass term, starting from eq. (11) as zero approximation function (where u is now an ordinary Dirac spinor), and taking into account, as in the usual theory, the negative energy states, is given in the second order of perturbation theory by the formula

$$\triangle m = 1.037 \int_1^\infty G^2 \left(0.0029 \frac{\varepsilon^2 - 1}{\varepsilon}\right) \frac{3\varepsilon^2 - 1}{\varepsilon^3} \, d\varepsilon \times 10^{-30} \text{g}, \tag{17}$$

where

$$G(\rho) = 3 \sqrt{\frac{\pi}{2}} \rho^{-\frac{3}{2}} J_{\frac{3}{2}}(\rho). \tag{18}$$

By numerical integration one obtains $m_e = m'' + \Delta m = 9.115 \times 10^{-28}$ g with an error in the numerical calculation which is certainly less than 0.04 % of the total mass. The experimental value is $(9.1083 \pm 0.0003) \times 10^{-28}$ g.

5. The Strangeness Quantum Number

A satisfactory theory for elementary particles must give an explanation of the existence of the strangeness quantum number and of the validity of the Gell-Mann—Nishijima scheme; the known classification of the elementary particles must be related with the fundamental laws of the theory.

We begin by showing here the existence in our formalism of a quantum number which can be interpreted as the strangeness; this interpretation will be extended and completed in the following two sections.

Let us consider the internal angular momentum tensor $J_{\lambda\mu}$ which, in terms of the coordinates of the internal pseudoeuclidean space, can be

expressed by the formula

$$J_{\lambda\mu} \equiv i\left(\eta_\mu \frac{\partial}{\partial \eta_\lambda} - \eta_\lambda \frac{\partial}{\partial \eta_\mu}\right). \tag{19}$$

It is well known from the theory of the representation of the homogeneous Lorentz group [5]) that the three operators

$$\mathbf{M}^2 = J_{23}^2 + J_{31}^2 + J_{12}^2,\ M_3 = J_{12} \quad \text{and} \quad \mathbf{N}^2 = -J_{41}^2 - J_{42}^2 - J_{43}^2 \tag{20}$$

can be simultaneously diagonalized. In section 2 we have discussed the connection between the first two operators and the masses. We shall now see that $\mathbf{N}^2$ can be very closely connected with the strangeness (cf. ref.[2f])). $\mathbf{N}$ is related to $\mathbf{M}$ by the relations

$$\mathbf{N}^2 = -2F + \mathbf{M}^2, \tag{21}$$

$$\mathbf{M} \cdot \mathbf{N} = G, \tag{22}$$

where F and G are invariant with respect to the transformations of the homogeneous Lorentz group. The $\mathbf{N}^2$ operator can be expressed in terms of the internal coordinates by the formula

$$\mathbf{N}^2 = -\eta_0^2 \Delta_\eta - \frac{\partial}{\partial \eta_0} \sum_1^3{}_k \eta_0 \eta_k \frac{\partial}{\partial \eta_k} - \sum_1^3{}_k \eta^2{}_k \frac{\partial^2}{\partial \eta_0^2} - 3\eta_0 \frac{\partial}{\partial \eta_0} - 2\eta_0 \sum_1^3{}_k \eta_k \frac{\partial^2}{\partial \eta_k \partial \eta_0}$$
$$(\eta_0 = -i\eta_4). \tag{23}$$

Then the $\mathbf{N}^2$ eigenvalues depend on η_0 and are thus arbitrary, because in the rest system of the particle, as is known from section 2, one has $\eta_0 = A$, where A is an arbitrary real constant. A further condition is then necessary to fix $\mathbf{N}^2$ and at the same time η_0 or the invariants G and F.

In the theory of the representation of the homogeneous Lorentz group the invariants G and F can be expressed as follows [5]):

$$2F = j_0^2 - 1 - \nu^2, \tag{24}$$

$$G = j_0 \nu, \tag{25}$$

where j_0 is the minimum j value of the j-representation, i.e. $j_0 = 0$ for bosons and $j_0 = \frac{1}{2}$ for fermions and ν is an arbitrary constant. In order to have an hermitian representation of the homogeneous Lorentz group, ν must always be real for $j_0 \neq 0$, while for $j_0 = 0$ it can be purely imaginary provided that ν^2 satisfies the condition

$$\nu^2 \geqq -1 - j(j+1) \tag{26}$$

which insures that $\langle \mathbf{N}^2 \rangle$ is positive, as it is immediately seen from the equation

$$\langle \mathbf{N}^2 \rangle = -j_0^2 + 1 + \nu^2 + j(j+1). \tag{27}$$

Now we introduce the following condition:

$$\nu^2 = \pm(2n-1)^2+B, \tag{28}$$

where the positive or negative sign corresponds to a Bessel function of positive or negative order respectively and $n(= 1, 2, \ldots)$ is the position number of the zero corresponding to the considered mass in the increasing succession of all zeros of the $J_{k+\frac{1}{2}}$ Bessel function of the particle (with $k = j$ in the case of bosons and $k = j-\frac{1}{2}$ in the case of fermions). The constant B is determined in such a way that the $\mathbf{N}^2$ eigenvalues of the pion and of the nucleon are zero. Then, as can be readily seen, one gets $B = 0$ for mesons and $B = -\frac{53}{2}$ for baryons.

One can thus formally define an orthogonal system of eigenvectors of the $\mathbf{N}^2$ operator corresponding to the eigenvalues given by condition (28). Owing to the fact that in the rest system the eigenvalues of the $\mathbf{N}^2$ operator are essentially determined (cf. eq. (23)) by the term $-\eta_0{}^2\Delta_\eta \to \eta_0{}^2 m_n{}^2$, η_0 must be formally replaced by an operator whose eigenvalues η_0 determine the eigenvalues of $\mathbf{N}^2$.

We meet here the difficulty that in the case of baryons ν can be purely imaginary (as follows from eq. (28)) and then a hermitian representation of the homogeneous Lorentz group cannot be obtained. We shall see in section 6 that by assuming an indefinite metric for the eigenvectors of the $\mathbf{N}^2$ operator, a pseudohermitian representation can be obtained.

Condition (28) relates the $\mathbf{N}^2$ eigenvalues to the fundamental law $J_{k+\frac{1}{2}}(ml) = 0$. Although at the present stage this condition may appear somewhat arbitrary, we shall see in this section and in section 6 that on its basis the correct value of the strangeness of the different elementary particles can be obtained and at the same time all unobserved particles listed in tables 1 and 2 can be excluded as possible products of a π+N reaction, because strangeness is not conserved or by considerations related with the pseudohermitian representation implied by eq. (28); moreover many fundamental interesting consequences concerning the mathematical structure and the physical interpretation of the theory can be deduced from it.

Using condition (28) one can see that the following relation holds between $\mathbf{N}^2$ and the attribute A (or strangeness $S = -A$) of the usual classification of elementary particles:

$$A = \pm\sqrt{\tfrac{1}{8}\langle\mathbf{N}^2\rangle}. \tag{29}$$

The attribute is thus proportional to the modulus of the $\mathbf{N}$ vector †.

† *Note added in proof:* The strangeness and the related conservation law for a many-particle system will be formulated in a rigorous covariant form in a further paper (submitted for publication in Acta Phys. Polonica). The additive character of the strangeness will be automatically derived from the theory and present results will be considered from a more systematic point of view. The connexion between strangeness and parity non-conservation will be seen to find a very natural place in the present theory.

To explain the additive character of A we note that the direction of the **N** vector is completely undetermined because the components of **N** cannot be diagonalized. Then, without contradicting the theory one can assume that the **N** vectors of all particles are parallel in the η space and the **N** vectors of all antiparticles antiparallel to those of particles, since with this assumption the direction of the **N** vectors is still undetermined. Thus, defining A as proportional to the modulus of the **N** vector of a system of particles and antiparticles, it must be expressed as the algebraic sum of the moduli of the **N** vectors of the single particles and antiparticles of the system.

Now, one has the fundamental fact that conservation of the modulus of the **N** vector, and consequently of A, in the π+N reaction leads to the exclusion as possible products of the reaction of all particles listed in tables 1 and 2, with the only exception of the particles listed in tables 3 and 4, i.e. with the exception of the observed particles and of two other spinorial particles with masses 1 578 m_e and 2 860 m_e and attributes $A = 0$ and $A = 2$, respectively. Indeed, the matrix element of an interaction commuting with $\mathbf{N}^2$ must vanish when it corresponds to initial and final states with different strangeness, owing to the orthogonality of the $\mathbf{N}^2$ eigenstates.

The 1 578 m_e and 2 860 m_e particles will be excluded in section 7 as physically observable particles by selection rules related with the indefinite metric of the eigenvectors of the $\mathbf{N}^2$ operator.

TABLE 3

Bosons

Mass	n	j	$\mathbf{N}^2$
265	1	0	0
972	1	2	8

TABLE 4

Fermions

Mass	n	j	$\mathbf{N}^2$
1 578	1	$\frac{3}{2}$	0
1 839	3	$\frac{1}{2}$	0
2 185	2	$\frac{3}{2}$	8
2 600	2	$\frac{13}{2}$	32
2 310	3	$\frac{5}{2}$	8
2 860	4	$\frac{5}{2}$	32

To conclude this section we note the following regularity (cf. ref.[2c]) and tables 1 and 2). Let us assume that the internal intrinsic parity of an

internal function (i.e. the intrinsic parity with respect to a reflection in the η space) be equal to $-(-1)^r$ where r is the position number of the mass in the increasing succession of all masses obtainable with the mass quantization condition (10) for all positive half-odd order Bessel functions (or of the increasing succession of all masses obtainable with all negative half-odd order Bessel functions, if the mass considered is a zero of a negative half-odd order Bessel function, as the pion). Then, by passing in tables 1 and 2 from one order of the Bessel function to the next higher order the internal intrinsic parity of the observed particles becomes even or odd alternatively.

The above assumption concerning the internal intrinsic parity of the internal function will be of fundamental interest for the developments of the following section, where a definite physical meaning will be attached to the concept of internal intrinsic parity.

6. The Connection between Masses and Isobaric Spin

We are now in a position to discuss the intimate connection between the masses and the isobaric spin and at the same time the deep physical meaning of the internal space.

First we note that if φ is a solution of the fundamental internal equation (7) the function

$$\varphi' \to a\varphi + b\gamma_5\varphi^c \qquad (|a|^2+|b|^2 = 1) \tag{30}$$

obtained by a Pauli canonical transformation of φ is also a solution of equation (7). This is a consequence of the fact that, in contrast to the Dirac equation, the mass term of eq. (7) does not contain the matrix β.

Eq. (7) allows also the transformation

$$\varphi' \to \exp(i\alpha\gamma_5)\varphi \qquad (\alpha \text{ real}). \tag{31}$$

We note that the same transformation properties hold for the internal equation describing the interaction of φ with the electromagnetic field, obtained in the rest system from eq. (5).

Now, we make the assumption (indicated with A) that only the group of transformations in the three-dimensional internal η space of the particles can be physically performed, which group is simply isomorphic to the group of canonical transformations (30) or (31) of the internal function φ of the particles.

In order to find the physical consequences and the meaning of the above assumption let us investigate which transformation group in the η space is isomorphic with the group of canonical transformations of φ.

Inserting in the Pauli canonical transformation (30) the one-column †

† The following representation for the γ matrices is used:

$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ $(i = 1, 2, 3)$, $\gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; σ_i Pauli matrices, $\varphi^c = C\bar{\varphi}^T$, $C = i\gamma_1\gamma_3$.

representation of the spinor φ, one finds that the following unimodular unitary transformations are induced by (30) for the four components φ_i:

$$\begin{aligned} \varphi_1' &\to a\varphi_1 - b\varphi_4^* \\ \varphi_4^{*\prime} &\to b^*\varphi_1 + a^*\varphi_4^*, \end{aligned} \tag{32}$$

$$\begin{aligned} \varphi_2' &\to a\varphi_2 + b\varphi_3^*, \\ \varphi_3' &\to -b\varphi_2 + a^*\varphi_3^* \qquad (|a|^2+|b|^2 = 1). \end{aligned} \tag{33}$$

As is well known [6]), to each unimodular unitary transformation (32) or (33) corresponds a rotation in the three-dimensional euclidean space and by virtue of this correspondence a representation of the group δ_3 of proper rotations can be obtained by the group u of transformations (32) or (33). This representation is double-valued, for on multiplying a and b by -1 the normalization condition $|a|^2+|b|^2 = 1$ is unaffected.

We now add to the transformation group u the transformations

$$\varphi_i' \to -\varphi_i \tag{34}$$

and postulate that to these transformations a reflection in a three-dimensional euclidean space must correspond. Then to the multiplication of a and b by -1 in (32) or (33) a reflection in a three-dimensional space is associated and a single-valued representation of the augmented rotation group δ'_3 including reflections can be obtained by the enlarged transformation group u' obtained by adding to u the transformations (34). The groups δ'_3 and u' are simply isomorphic.

Now let us assume that the three-dimensional euclidean space is the η space on which φ depends. Then, only solutions φ with odd internal intrinsic parity can be the elements undergoing the transformation group u' including the transformations (34). From assumption A it follows that any transformation of the augmented group δ'_3 of rotations can be physically performed in the internal space of a particle described by a φ with odd internal intrinsic parity.

This is not the case for particles described by a φ with even internal intrinsic parity which cannot undergo the transformations (34). However, in this case isomorphism holds between a subgroup of δ'_3 and the group of the second canonical transformations (31). Indeed one finds that the following transformation is induced for the components φ_i by (31):

$$\varphi'_1 \to (\cos\alpha + i\sin\alpha)\varphi_1, \tag{35a}$$

$$\varphi'_3 \to (\cos\alpha - i\sin\alpha)\varphi_3 \tag{35b}$$

and analogous formulas for the indices 2 and 4. As is well-known, the group of linear transformations (35a) or (35b) with $0 \leqq \alpha < 2\pi$ is isomorphic with the group of rotations in a plane, with α as angle of rotation. Then isomorphism holds between the group of canonical transformations (31) with

$0 \leqq \alpha < 2\pi$ and the subgroup δ_2 of rotations in η space around the z-axis, and from assumption A it follows that only this subgroup δ_2 of rotations can be performed in the internal space of a particle described by a φ with even internal intrinsic parity.

Now the $2j+1$ solutions φ of the internal equation (7) or (8) corresponding to the $2j+1$ M_3-eigenstates of a particle with given mass and j, are the substratum of a reducible $(2j+1)$-dimensional representation of the subgroup δ_2 of rotations about the z-axis. If one requires that the set of physical solutions of the internal equation (7) form the substratum of an *irreducible* representation of the group of transformations which can be performed in the internal η-space according to the assumption A, one obtains that in the case of a φ with even internal intrinsic parity, the $(2j+1)$-dimensional representation must be reduced to the irreducible one-dimensional representation of δ_2, whose substratum is only *one* solution belonging to *one* eigenvalue of M_3. Thus, if only the system of solutions φ is allowed which form the substratum of an irreducible representation of the group of the physical transformation in η space, we conclude that *one* M_3 state only is admitted in the case of a non interacting particle described by a φ with even internal intrinsic parity. This excludes the existences of "bare" particles with the same mass and j, but different M_3 or a solution which is a superposition of the different M_3-eigenstates, according to the fact that, owing to assumption A, the particles cannot be found in rotated position of **M** with respect to the z-axis.

It is very important to note that assumption A concerns the internal function φ of a bare particle only and the previous considerations and conclusions cannot be applied to the solution $\psi(x, \eta)$ describing an interacting particle. For this ψ the transformation properties (30) and (31) do not generally hold because a mass term due to the interaction with the internal function must be added to the equation describing the observable motion, as we done in the case of the electron. Thus the previous conclusions concern *bare* particles only.

The above results provide the key for understanding the connection between masses and isobaric spin. From table 2 one sees that in the case of the bare nucleon, the bare Λ particle and the bare 1578 m_e particle one value only of the third component M_3 of the total internal angular momentum is possible.

Let us define the third component T_3 of the isobaric spin of a bare elementary particle by the equation

$$T_3 = M'_3 - M''_3 + \begin{cases} \frac{1}{2}\tau_3 \text{ for isofermions } (\tau_1, \tau_2, \tau_3\text{: Pauli spin matrices}), \\ 0 \text{ for isobosons}, \end{cases} \tag{36}$$

where M'_3 and M''_3 correspond to the *only possible* eigenvalues of the third

component of the total internal angular momentum operator **M** of the non-interacting particle with given mass and j. The isofermion or isoboson character of a particle will be deduced below.

To illustrate the above definition we note that from equation (36) it follows that if a non-interacting particle can be in a state with, say, $M_3 = 0$ and its isobaric spin is $T = 1$, it can assume only the states with $M_3 = \pm 1$. Any other state of M_3 is impossible. Thus if $T \leqq 1$, and if one assumes that both symmetric states $\pm M_3$ are possible (if a particle with odd internal parity is concerned) and moreover if one assumes that a free particle can be always found in a state $M_3 = 0$ or $M_3 = \pm\frac{1}{2}$ (i.e. one of the values $M_3 = 0$, $+\frac{1}{2}$, $-\frac{1}{2}$ must always be contained in eq. (30)), the degeneration of a free particle with respect to M_3 can be restricted. Indeed, if $T = 0$ or $T = \frac{1}{2}$, the only possible M_3 states of the particle must be $M_3 = 0$, or $M_3 = \pm\frac{1}{2}$; in the case $T = 1$ the two possible states are $M_3 = \pm 1$ for a boson or $M_3 = \pm\frac{1}{2}$ for a fermion. The states $|M_3| \geqq \frac{3}{2}$ are excluded because, if the particle can be, for example, in the state $M_3 = +\frac{3}{2}$, it cannot be in the state $M_3 = -\frac{3}{2}$, contrary to the assumption.

In the case of a bare particle with even intrinsic internal parity, one value only of the third component of the total internal angular momentum is possible; then, from eq. (36) we have for the isobaric spin the two possibilities $T = 0$ or $T = \frac{1}{2}$. In the case of a particle with odd intrinsic internal parity, T_3 is not determined by eq. (36) owing to the degeneration of **M** with respect to the orientation. To determine T_3 we add the further condition that a linear homogeneous relation must hold for all particles between the attribute A, the third component T_3 of the isobaric spin, the electric charge Q (in units of e) and the baryonic number N:

$$xA + yQ + zT_3 + N = 0, \tag{37}$$

where the coefficients x, y, z are unknown for the moment, but certainly different from zero; as only the ratios between the coefficients are of interest, we have put equal to 1 the coefficient of the baryonic number N.

In the case of strong interactions, which does not induce rotations with respect to the z-axis in the internal space of the system of particles, T_3 and A, as well as Q and N, are constants of the motion and a linear relation certainly holds between them; as a consequence of the additivity of A, T_3, Q and N of the different particles of a system, this relation must be homogeneous. Now T_3 and x, y, z can be determined by the requirement that the same relation (37) holds for general interactions which induce rotations of the system of particles with respect to the z-axis in the internal space, and are such that T_3 and A are not constants of the motion of the system of particles. Thus, the above requirement enables a definition of T_3 even in the case of an *interacting* particle for which assumption A does not hold and the third

component of the internal angular momentum cannot be diagonalized and is not a constant of the motion, so that the definition (36) has no meaning.

Let us consider first the nucleon: we know that only the two cases $T_3 = 0$ and $T_3 = \pm\frac{1}{2}$ are possible; however, the former case is excluded because, putting in eq. (37) $A = 0$, as we know from section 5, one would obtain $N = 0$; then we are left with $T_3 = \pm\frac{1}{2}$ and from eq. (37) one has

$$\begin{aligned} y+\tfrac{1}{2}z+1 &= 0 \text{ for the proton,} \\ -\tfrac{1}{2}z+1 &= 0 \text{ for the neutron,} \end{aligned} \tag{38}$$

and the values $y = -2$, $z = 2$ are deduced. Also the Λ particle can have $T_3 = 0$ or $T_3 = \pm\frac{1}{2}$; however, the latter case leads to $x = 0$ and must be excluded; from $T_3 = 0$ and from eq. (37) one obtains

$$x+1 = 0. \tag{39}$$

Thus, relation (37) becomes

$$-A = 2(Q-T_3-\tfrac{1}{2}N) \tag{40}$$

which is the fundamental relation of the Gell-Mann-Nishijma scheme. From this relation the T_3 values of all other particles can be deduced from the corresponding values of A obtained in section 5.

The surprising consequence of our formalism is that the isobaric spin of the pion must be zero because the pion has an internal angular momentum equal to zero.

The existence of a pion with zero isobaric spin does not seem to be contradicted by experiment [7]), but we know that pions with $T = 1$ also exist. Our result seems to indicate that the existence of charged pions is not due to an isotopic spin different from zero, but to the interaction of the neutral pion with a charged particle which gives rise to a bound system having the properties a charged pion. Thus, if the bound system has $M_3 = \pm 1$, the neutral pion with $T_3 = 0$ and the charged bound system can be kept together to form a triplet with $T = 1$ (see appendix). A non-direct electromagnetic interaction is in accordance with the fact that if one calculates the self-energy of a charged pion under the assumption of a direct interaction with the electromagnetic field (introducing the electromagnetic interaction with bosons in the usual way and assuming the dependence of the electromagnetic field on $x+\eta$) one finds, by the perturbation method, that the electrostatic self-energy diverges logarithmically while the quadratic divergence due to the fluctuation of the vacuum (the quadratic term of the interaction) remains unaltered [8]). The divergence of the electrostatic self-energy is due to the fact that the cut-off of the high momentum photons, which is an automatic consequence of the theory, is not sufficiently strong owing to the singularity at the origin of the internal function $\cos(m_\pi r)/r$

of the pion. Then, if the bilocal theory is valid and if the divergency does not depend on the perturbation method, we must conclude that a pion with finite mass directly interacting with the electromagnetic field cannot exist.

In the appendix an attempt will be presented to explain the existence of pions with $T = 1$, which, at the same time, enables an explanation of the muons.

7. The Norm of Baryons

In this section we discuss the non-hermitian representation of the homogeneous Lorentz group which, in the case of baryons, is implied by the definition (28) of the modulus of the **N** vector. For $n \geqq 4$ one has a real ν and the representation is hermitian. For $n < 4$, ν is purely imaginary and we can put $\nu = i\nu'$ where ν' is real. On the other hand, since $G = \mathbf{M} \cdot \mathbf{N} = \langle j|N|j\rangle j(j+1) = j_0\nu$, one can put $G = iG'$, $\mathbf{N} = i\mathbf{N}'$ and from (21) and (22) one has the following two relations:

$$2F = \mathbf{M}^2+\mathbf{N}'^2, \tag{41}$$

$$G' = \mathbf{M} \cdot \mathbf{N}', \tag{42}$$

where $2F$ and G' are invariant with respect to the rotation group in an *euclidean* four-dimensional space. Let us now assume that the $\mathbf{N}'^2$ eigenstates have negative norm; then,

$$2F = \langle\mathbf{M}^2\rangle-\langle\mathbf{N}'^2\rangle, \tag{43}$$

$$G' = \mathbf{M} \cdot \mathbf{N}', \tag{44}$$

and $2F$ and G' are invariant with respect to transformations of the homogeneous Lorentz group (in the *pseudoeuclidean* four-dimensional space). Thus, a representation of the homogeneous Lorentz group can be obtained from a representation of the rotation group in the euclidean four-dimensional space by assuming a negative norm of the $\mathbf{N}'^2$ eigenstates. Putting (cf. ref. [5]), p. 24)

$$\Psi(j) = \langle j|N'|j-1\rangle\langle j-1|N'|j\rangle(2j-1)(2j+1), \tag{45}$$

and taking into account the negative norm of the $\mathbf{N}'^2$ eigenstates (by changing the original sign of all matrices with respect to $\mathbf{N}^2$) one obtains with some calculation

$$\Psi(j) = -(2F+1)+\frac{G'^2}{j^2}+j^2. \tag{46}$$

As in the usual theory of the representation of the Lorentz group, the condition $\Psi(j) \geqq 0$ and the reality of $2F$ and G' insure that the matrix

representation of the **M** and **N** operators is formally hermitian. In the present case, when eigenvectors of negative norm exist, we call this representation a "pseudohermitian" representation.

We have then arrived at the fundamental conclusion that all baryons with $n < 4$ are described by an eigenvector of $\mathbf{N}'^2$ which has a negative norm, thus, an indefinite metric is the necessary framework for the present theory.

The 2 860 m_e ($n = 4$) particle has positive norm and this fact enables us to exclude this particle as a possible physical product of a $\pi+\mathrm{N} \to 2\,860\, m_e + 2\mathrm{K}$ reaction. Indeed, it is known from the work of Ascoli and Minardi [9]) on the physical interpretation of a theory with indefinite metric that transition probabilities cannot be defined if the initial and final states do not have the same norm. Then, if one starts from a state with a negative norm, as a $\pi+\mathrm{N}$ state, the 2 860 $m_e+2\mathrm{K}$ state cannot be included in the physical states for which a unitary scattering matrix can be defined, and the 2 860 m_e particle cannot appear as a physical state: on the other hand, the production of this state with positive norm is automatically excluded by the formalism. Indeed, owing to the fact that in the strong interaction the strangeness is conserved, the interaction hamiltonian must commute with the $\mathbf{N}^2$ operator. Thus, the matrix element for the production of states with different norm with respect to the norm of the initial state is zero because the $\mathbf{N}^2$ eigenstates of positive and negative norm must be orthogonal (see section 5).

Let us now obtain the j values for which condition $\Psi(j) \geqq 0$ is satisfied. Using the equations $2F = j_0^2 - 1 + \nu'^2$, $G' = j_0\nu'$ one gets from eq. (46)

$$\Psi(j) = -j_0^2 - \nu'^2 + \frac{j_0^2\nu'^2}{j^2} + j^2 \geqq 0. \tag{47}$$

This condition is satisfied for $j^2 \geqq \nu'^2$, $j^2 \geqq j_0^2$. Now, $\nu'^2 = -\nu^2$, where ν^2 is given by the eq. (28). One easily verifies that condition (47) holds for the nucleon ($j = j_0$, $\Psi(j) = 0$), the Λ particle ($\nu'^2 = 17.5$; $j^2 = \frac{81}{4} = 20.25$), the Σ particle ($\nu'^2 = 1.5$; $j^2 = \frac{25}{4} = 6.25$) and the Ξ particle ($\nu'^2 = 17.5$: $j^2 = \frac{169}{4} = 42.25$) but not for the 1 578 m_e particle ($\nu'^2 = 25.5$; $j^2 = \frac{81}{4} = 20.25$). Thus, this particle is excluded because only a hermitian representation of the Lorentz group must be used.

A striking feature of the theory is that the numerous unobserved particles predicted in tables 1 and 2 are eliminated and only the observed particles remain: this is a consequence of the "magic" condition (28).

8. Conclusion

We were here only concerned with a first quantization problem and not with the second quantization or with a field theory. The different particles

are not the product of an interaction but of a generalization of the fundamental equations describing free particles. Although the common idea that particles can be derived from a single interaction is at first sight certainly attractive, we believe that elegance and simplicity are also reached by the present formulation.

A great number of problems of mathematical and interpretative nature now arise, which are essentially connected with the non-local character of the theory, the introduction of a universal length and the use of an indefinite metric. For example, the causality problem is highly important, and it must be noted that in the present theory it is put on an entirely new basis with respect to the conventional non-local theories, as our fundamental equations are not integro-differential equations but formally local linear differential equations in an eight-dimensional space. To stress this essential feature of our theory, in previous work we have called it "bilocal" (following Rayski) instead of "non-local".

The extension of our results to the case of a field-theory is also a fundamental problem whose solution is connected with the clarification of the physical and interpretative content of the bilocal hypothesis. With regard to this we note that a completely new feature of the present theory with respect to the usual local theory is represented by the supplementary condition (6) (which in a previous work [2c]) was called "no-coupling condition" because it expresses the separation between the two problems of the internal structure and of the observable motion of a particle †); indeed, this condition implies a definition of the internal three-dimensional space (intersection with the hyperplane $\eta_4 = iA$) in the rest system of the particle, and thus depending on the state of the particle. We have here a kind of interaction between the state and the particle whose deep physical meaning is rather unclear, but it is certainly in contradiction with the concept of field operator in a quantum theory with definite metric, because any operator corresponds to a quantity which may be measured whatever the state of the system is.

We hope that the present exposition of our bilocal theory will stimulate the attention of the physicists on these and other still open important questions.

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† We note that, owing to condition (6), the gauge invariance holds only if we restrict the gauge to the η variable. This compels us to choose a solution having the form $\psi = \exp(ipx)\varphi(\eta)$, because with this solution the four-current $s_\nu = \psi^* \alpha_\nu \psi = \varphi^*(\eta)\alpha_\nu \varphi(\eta)$ depends only on η, and the restricted gauge is sufficient to ensure four-current conservation.

Appendix

ATTEMPT AT THE EXPLANATION OF THE CHARGED PIONS AND MUONS

We introduce two interactions of the internal functions of the electron-neutrino pair with the internal function of the pion and of the muon and their coupling constants are assumed to be equal; then we can determine the common value of the coupling constants in such a way that the masses of $\pi_{\pm}$ and μ are predicted.

Concerning the internal functions only, these interactions cannot give rise to a decay of the pion or of the muon or to any effect in the motion of these particles in the observable space (scattering processes) but only to a contribution to the mass due to structure.

We neglect in what follows any effect due to the electric charge; then from section 4 one deduces that the internal function of the electron is given simply by the internal function of a Majorana neutrino. We assume the following pseudovector interaction density between the pion internal function and the internal functions describing the electron and the neutrino:

$$H' = \frac{g}{m_0}\left[\varphi_e{}^*\gamma_4\gamma_5\gamma_\alpha\left(\frac{\partial}{\partial\eta_\alpha}\varphi_\nu\right)\varphi_\pi + \left(\frac{\partial}{\partial\eta_\alpha}\varphi_\nu\right)^*\gamma_\alpha\gamma_5\gamma_4\varphi_e\varphi_\pi\right] \quad (\gamma_5{}^2 = 1). \quad (48)$$

This is the simplest interaction one can get because a pseudoscalar or a scalar interaction vanishes identically in the case of Majorana spinors.

Putting $\eta_4 = 0$, $\partial\varphi/\partial\eta_4 = 0$ (cf. section 2) in the rest system of the interacting pion, one obtains the following internal equation for φ_π:

$$\Delta_\eta\varphi_\pi + m^2\varphi_\pi = \frac{g}{m_0}\left[\varphi_e{}^*\gamma_4\gamma_5\boldsymbol{\gamma}\cdot\nabla_\eta\varphi_\nu + (\nabla_\eta\varphi_\nu)^*\cdot\boldsymbol{\gamma}\gamma_5\gamma_4\varphi_e\right]. \quad (49)$$

The interaction term is different from zero only in the case of parallel spins of the electron and the neutrino. Then the third component of the internal angular momentum of the system formed by the neutral pion and the electron-neutrino pair must be $M_3 = +1$ or $M_3 = -1$. From eq. (49) one obtains

$$\Delta_\eta\varphi_\pi + m^2\varphi_\pi = \pm\frac{g}{\frac{2}{3}\pi l^3} \quad (50)$$

according as the spins of the electron and the neutrino have a parellel or an antiparallel orientation to the direction of $p_3 = p_3{}^{(\eta)} + p_3{}^{(x)}$.

Assuming that the part of φ_π which satisfies the homogeneous equation corresponding to (50) has the same sign as the part which satisfies the inhomogeneous equation, we find that the neutral pion internal function interacting with the internal function of the electron-neutrino pair is

$$\varphi_\pi(mr) = \pm A\sqrt{\frac{m}{2\pi l}}\frac{\cos mr}{mr} \pm \frac{3g}{2\pi l^3 m^2}, \quad (51)$$

where A is a positive constant of normalization and can be put $= 1$, if the second term is very small as compared with the first term (as will be the case)

By means of the mass quantization condition $\varphi_\pi(ml) = 0$, one obtains from eq. (51) the following equation which is independent of the mass m_0 of the neutrino:

$$x^{\frac{3}{2}} \cos x = -\frac{3g}{\sqrt{2\pi}}, \qquad (x = ml). \quad (52)$$

The lowest root of eq. (52) must give the mass of π when its internal function interacts with the internal function of the electron-neutrino pair. Eq. (52) has no real negative roots.

We now deal with the interaction of a spinor internal function and the internal function describing the internal structure of an electron-neutrino pair. Let us assume that the spins of the pair are antiparallel, so that the system composed by this pair is a boson with spin zero and mass m_e. Then the internal function describing the structure of the pair will be a scalar or a pseudoscalar function $\varphi_{e\nu}$ given by the formula

$$\varphi_{e\nu} = \frac{1}{\sqrt{m_e}} f(\eta_\alpha{}^2) \exp[-i\eta_\alpha p^{(\eta)}_{\alpha e\nu}], \quad (53)$$

where m_e is the electron mass and $f(\eta_\alpha{}^2)$ is normalized within a sphere with radius l and describes, as is known from section 4, the internal structure of the neutrino and the electron.

We assume the following pseudovector interaction density between the internal function of the spinor particle φ_μ and $\varphi_{e\nu}$:

$$H' = \frac{g}{m_\mu}\left[\varphi_\mu{}^* \gamma_4\gamma_5\gamma_\alpha \left(\frac{\partial}{\partial\eta_\alpha}\varphi_\mu\right)\varphi_{e\nu} + \left(\frac{\partial}{\partial\eta_\alpha}\varphi_\mu\right)^* \gamma_\alpha\gamma_5\gamma_4\varphi_\mu\varphi_{e\nu}\right], \quad (54)$$

where g is the same coupling constant as that of the interaction (48).

We now calculate approximately the contribution of the interaction (54) to the rest energy of the spinor particle by substituting in a first approximation to φ_μ in eq. (54) the internal function of a Majorana neutrino with mass $m_0 = m_\mu = p^{(\eta)}_{3\mu} = p^{(\eta)}_{0\mu}$. By putting $\eta_4 = 0$, $\partial\varphi/\partial\eta_4 = 0$ (cf. section 4) in the rest system of the interacting muon and integrating (54) on the volume of the sphere, one obtains, in the limit $p^{(\eta)}_{3e\nu} \to 0$, the following expression for the rest energy:

$$m = m_0 + \langle H' \rangle = m_0 + \frac{ig}{m_0} u_p{}^* \gamma_5\gamma_4\gamma_3 p^{(\eta)}_{3\mu} u_p \frac{2}{\sqrt{\frac{4}{3}\pi l^3 m_e}} \xrightarrow{m_0 \to 0} \frac{g}{\sqrt{\frac{1}{3}\pi l^3 m_e}}. \quad (55)$$

On reversing the spin direction in the matrix u_p a change of sign in the mass is obtained.

Let us determine the coupling constant g in such a way that, for the value $l = 2.30 \times 10^{-13}$ cm, eq. (55) yields the mass 206 m_e of the μ meson. Thus, $g = 1, 11e$, where e is the electronic charge, and from equation (52) the value 273.5 m_e is obtained for the mass of the π interacting with the internal functions of the electron-neutrino pair.

The above procedure has the disadvantage that *ad hoc* interactions are introduced, but the numerical value obtained is very accurate.

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RELATIVE TRANSITION PROBABILITIES BETWEEN ROTATIONAL LEVELS OF NON-AXIAL NUCLEI

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Abstract: Energies and wave functions are calculated for the rotational states ($J = 4, 6, 8$) of non-axial nuclei. Relative reduced E2 transition probabilities for the rotational band levels of such nuclei are derived. Conditions under which the rotational states can be characterized by quantum numbers K are established. It is shown that if the nuclear shape differs from axial symmetry, the 1 : 3.3 : 7 : 12 interval rule observed in rotational bands of axial nuclei is violated. The theory is compared with the experimental data.

1. Introduction

Rotational levels in even nuclei were investigated by Davydov and Filippov [1]) under the assumption that in a first approximation the equilibrium shape of the nucleus can be represented by a triaxial ellipsoid. Analytical expressions for the rotational level energies were derived for spins of 2, 3 and 5, and the transition probabilities between these levels were computed. It was demonstrated, in particular, that the theory yields a unique relation between the ratio of the energies of two spin 2 levels and the ratio of the reduced transition probabilities from the second spin 2 level to the first level (cascade transition) or directly to the ground state (direct transition).

In the present paper some results of numerical calculations of the energies of rotational states with spins 4, 6, 8 are given as functions of the parameter γ which specifies the deviation of the nuclear shape from axial symmetry [2]). The wave functions of these excited states and the transition probabilities between them are calculated. In section 4 we define the conditions under which rotational states can be described by approximate wave functions corresponding to states with a definite total momentum on axis 3 of the nucleus. In the same section approximate expressions are derived for reduced E2 probabilities between rotational nuclear states which only slightly differ from axial symmetry. The theory is compared with the experimental data in section 5.

2. Energy of Excited States Possessing Spins of 4, 6 or 8

In the adiabatic approximation the rotational energy of a non-spherical even nucleus is defined by the Schroedinger equation:

$$(H-\varepsilon)\Psi = 0, \tag{2.1}$$

where ε is measured in $\hbar^2(4B\beta^2)^{-1}$ units and the operator H is defined by

$$H = \tfrac{1}{2}\sum_{\lambda=1}^{3} \frac{\hat{J}_\lambda^2}{\sin^2(\gamma-\frac{2}{3}\pi\lambda)}; \tag{2.2}$$

$\hat{J}_\lambda$ are projections of the total angular momentum operator on the axes of a coordinate system connected with the nucleus.

The wave function Ψ which corresponds to a state with a total angular momentum J and satisfies the symmetry conditions established by Bohr [2]) can be written down as

$$\Psi_{JM} = \sum_K |JK\rangle A_K, \tag{2.3}$$

where

$$|JK\rangle = \left[\frac{2J+1}{16\pi^2(1+\delta_{K0})}\right]^{\frac{1}{2}}\left[D^J_{MK}+(-1)^J D_{M,-K}\right]. \tag{2.4}$$

In (2.4) the D^J_{MK} are functions of the Eulerian angles specifying the orientation of the principal nuclear axes in space. It can be shown that wave functions (2.3) form the basis of a completely symmetric representation of the group $\mathscr{D}_2$ (see ref.[1])) whose elements are rotations through 180° about each of the three principal nuclear axes.

Inserting (2.3) in (2.1) and employing the values of the matrix elements of the rotational energy operator (2.2) on the wave functions (2.4), we obtain for each value of J a set of algebraic equations defining the values of the coefficients A_K in the wave functions (1.3). For example, for $J = 4$ the Schroedinger equation (1.1) reduces to the set of equations

$$\begin{gathered}
[5(\alpha+\beta)-\varepsilon]A_0+\tfrac{3}{2}\sqrt{5}(\alpha-\beta)A_2 = 0,\\
\tfrac{3}{2}\sqrt{5}(\alpha-\beta)A_0+[4(\alpha+\beta)+2\delta-\varepsilon]A_2+\tfrac{1}{2}\sqrt{7}(\alpha-\beta)A_4 = 0,\\
\tfrac{1}{2}\sqrt{7}(\alpha-\beta)A_2+[(\alpha+\beta)+8\delta-\varepsilon]A_4 = 0,
\end{gathered} \tag{2.5}$$

where

$$\alpha = \sin^{-2}(\gamma-\tfrac{2}{3}\pi), \qquad \beta = \sin^{-2}(\gamma+\tfrac{2}{3}\pi), \qquad \delta = \sin^{-2}\gamma.$$

The energies of the corresponding rotational states can be derived from the condition of solubility of such sets of equations. By solving each set (for corresponding values of ε) one can determine the wave functions of these states.

The wave functions (2.3) and the levels possessing spins 2, 3, 5 can be expressed in terms of parameter γ by analytical formulas. The latter are given in ref.[1]). For the energies of spins 4, 6 and 8 levels algebraic equations of respectively third, fourth and fifth degree with respect to ε are obtained. Thus, for example, the equation for the energy of spin 4 levels is

$$\varepsilon^3 - \frac{90\varepsilon^2}{\sin^2 3\gamma} + \frac{48[27+26\sin^2 3\gamma]\varepsilon}{\sin^4 3\gamma} - \frac{640[27+7\sin^2 3\gamma]}{\sin^4 3\gamma} = 0. \quad (2.6)$$

The explicit form of the equations for the energy of spin 6 and 8 levels has been presented in the previous paper of the authors [13]).

Results of numerical solution of these equations for several values of γ are given in table 1. The number in parenthesis after ε refers to the spin of the excited state and the subscript denotes the level number.

TABLE 1

Dependence of rotational energy (in $\hbar^2/4B\beta^2$ units) of even nucleus on parameter γ

γ^0	0	5	10	15	20	22.5	25	27.5	30
$\varepsilon_1(4)$	13.33	13.54	14.12	15.00	15.81	16.01	16.06	16.02	16
$\varepsilon_2(4)$	∞	274.1	77.73	42.48	32.26	30.92	31.21	32.69	34
$\varepsilon_3(4)$	∞	1056	268.1	122.5	71.92	58.51	49.20	42.85	40
$\varepsilon_1(6)$	28	28.42	29.51	30.65	30.78	30.51	30.33	30.06	30
$\varepsilon_2(6)$	∞	270.5	93.51	60.46	54.52	55.74	57.33	59.71	60
$\varepsilon_1(8)$	48	48.60	50.26	50.87	49.60	48.94	48.38	48.10	48

3. Quadrupole Electric Transition Probabilities in the Rotational Band

The reduced probability for electric quadrupole transitions between two states described by the functions Ψ_{Jm1} and $\Psi_{J'm't}$ can be described by the expression

$$B(\mathrm{E2};\mathrm{i}\to\mathrm{f}) = \frac{5}{16\pi^2(2J+1)}\sum_{\mu m' m}|(J'm'\mathrm{f}|\hat{Q}_{2\mu}|Jm\mathrm{i})|^2, \quad (3.1)$$

where

$$\hat{Q}_{2\mu} = eQ_0\left\{D^2_{\mu 0}\cos\gamma + \frac{\sin\gamma}{\sqrt{2}}(D^2_{\mu 2}+D^2_{\mu,-2})\right\}, \quad Q_0 = \frac{3ZR^2\beta}{\sqrt{5\pi}}.$$

The wave functions for states with spins of 2 and 3 have been computed in ref.[1]). The coefficients A_K for wave functions of states of spin 4

$$\psi_{4m1} = \sqrt{\frac{9}{8\pi^2}}\left\{A_{01}D^4_{m0} + A_{21}\frac{D^4_{m2}+D^4_{m,-2}}{\sqrt{2}} + A_{41}\frac{D^4_{m4}+D^4_{m,-4}}{\sqrt{2}}\right\} \quad (3.2)$$

can be evaluated by solving the set of equations (2.5) for each of the roots of the equation (2.6) which defines the energy values of these levels. Values of coefficients A_K for two spin 4 levels are presented in table 2. The values of the coefficients of the wave functions

$$\Psi_{6m1} = \sqrt{\frac{13}{8\pi^2}}\left\{B_{01}D^6_{m0} + B_{21}\frac{D^6_{m2}+D^6_{m,-2}}{\sqrt{2}} + B_{41}\frac{D^6_{m4}+D^6_{m,-4}}{\sqrt{2}} + B_{61}\frac{D^6_{m6}+D^6_{m,-6}}{\sqrt{2}}\right\} \quad (3.3)$$

for spin 6 states are also given in table 2.

TABLE 2

Coefficients defining wave functions (3.2) and 3.3) of states with spins of 4 and 6

γ^0	0	5	10	15	20	22.5	25	27.5	30
A_{01}	1	1	0.999	0.993	0.955	0.910	0.852	0.792	0.739
A_{21}	0	0.003	0.030	0.114	0.296	0.414	0.522	0.605	0.661
A_{41}	0	10^{-6}	10^{-4}	0.001	0.010	0.022	0.043	0.076	0.125
A_{02}	0	−0.003	−0.030	−0.114	−0.296	−0.415	−0.523	−0.602	−0.559
A_{22}	1	1	0.999	0.993	0.954	0.907	0.842	0.754	0.500
A_{42}	0	5×10^{-4}	0.004	0.015	0.043	0.074	0.128	0.267	0.661
B_{01}	1	1	0.998	0.973	0.878	0.817	0.766	0.714	0.672
B_{21}	0	0.0075	0.065	0.232	0.476	0.570	0.633	0.674	0.695
B_{41}	0	10^{-4}	6×10^{-4}	8×10^{-3}	0.043	0.081	0.113	0.189	0.254
B_{61}	0	10^{-7}	10^{-6}	6×10^{-5}	9×10^{-4}	2.8×10^{-3}	6.1×10^{-3}	0.015	0.031

By using (3.1) and the wave functions of states of spin 4 and 2 one can compute the reduced transition probabilities between these states. Thus, the reduced probabilities (in $e^2Q_0^2/16\pi$ units) for the quadrupole transition between spin 4 and spin 2 levels can be expressed by the formula

$$b(E2;\,4\mathrm{i}\to 2\mathrm{f}) = \tfrac{5}{126}\{\cos\gamma(6A_{01}a_\mathrm{f}+\sqrt{15}A_{21}b_\mathrm{f}) + \sin\gamma(\sqrt{15}A_{21}a_\mathrm{f}+A_{01}b_\mathrm{f}+\sqrt{35}A_{41}b_\mathrm{f})\}^2, \tag{3.4}$$

where a_f, b_f are coefficients specifying the wave functions of states possessing a spin of 2 (see ref. [1])).

The reduced probabilities for transitions between levels with a spin of 4 are given by the expression

$$b(\mathrm{E2};\,4\mathrm{i}\to 4\mathrm{f}) = \tfrac{1}{77}\{2\cos\gamma[7A_{41}A_{4\mathrm{f}}-5A_{01}A_{0\mathrm{f}}-2A_{21}A_{2\mathrm{f}}] + \sin\gamma[3\sqrt{15}(A_{21}A_{0\mathrm{f}}+A_{01}A_{2\mathrm{f}})+\sqrt{21}(A_{21}A_{4\mathrm{f}}+A_{41}A_{2\mathrm{f}})]\}^2. \tag{3.5}$$

Finally, the expressions for the reduced quadrupole electric transitions between levels with spins 4, 3 and 6, 4 are

$$b(\mathrm{E2};\,4\mathrm{i}\to 3) = \tfrac{1}{9}\{2\sqrt{3}\cos\gamma\cdot A_{21}+\sin\gamma[\sqrt{5}A_{01}-\sqrt{7}A_{41}]\}^2, \tag{3.6}$$

$$b(\mathrm{E2};\,6\mathrm{i}\to 4\mathrm{f}) = \tfrac{5}{143}\left\{\cos\gamma[3\sqrt{5}B_{01}A_{0\mathrm{f}}+4\sqrt{2.1}\,B_{21}A_{2\mathrm{f}}+3B_{41}A_{4\mathrm{f}}] + \sin\gamma\left[\sqrt{3}B_{01}A_{2\mathrm{f}}+\frac{B_{21}A_{4\mathrm{f}}}{\sqrt{10}}+\sqrt{14}B_{21}A_{0\mathrm{f}}+\sqrt{21}B_{41}A_{2\mathrm{f}}+\sqrt{49.5}B_{61}A_{4\mathrm{f}}\right]\right\}^2. \tag{3.7}$$

For the sake of brevity we shall call the energy levels 0, 21, 41, 61, 81 levels of the "ground rotational band". They are depicted in fig. 1 by solid lines. For $\gamma\to 0$ the levels of the ground rotational state go over to levels of an axially-symmetric nucleus. All other energy levels (depicted in fig. 1 by dashed lines) tend to infinity for $\gamma\to 0$. These energy levels will be called "anomalous" levels.

The reduced electric quadrupole transition probabilities between different rotational states of even nuclei can be calculated by using the wave function coefficients given in table 2 and ref.[1]). The values of some of the reduced probabilities are given in table 3. In the present paper reduced probabilities for all transitions are expressed in $e^2Q_0^2/16\pi$ units.

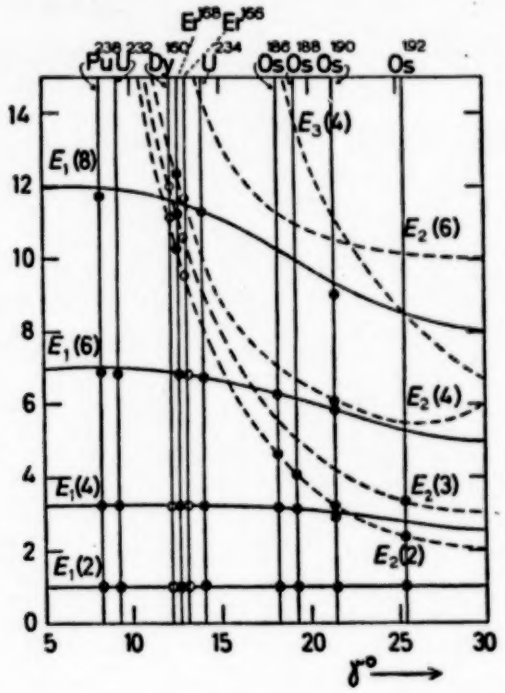

Fig. 1

TABLE 3

Reduced probabilities (in $e^2Q_0^2/16\pi$ units) for electric quadrupole transitions between some rotational states of an even nucleus

γ^0	0	5	10	15	20	22.5	25	27.5	30
b(E2; 3 → 41)	0	0.0060	0.034	0.130	0.406	0.619	0.821	0.955	1.00
b(E2; 41 → 21)	1.429	1.418	1.395	1.377	1.372	1.366	1.365	1.378	1.389
b(E2; 41 → 22)	0	3.5×10^{-4}	0.0023	0.010	0.033	0.044	0.039	0.016	0
b(E2; 42 → 21)	0	4.1×10^{-3}	0.011	0.008	4×10^{-4}	0.009	0.021	0.018	0
b(E2; 42 → 22)	0.595	0.591	0.575	0.543	0.481	0.447	0.435	0.484	0.595
b(E2; 42 → 3)	1.333	1.323	1.282	1.172	0.978	0.680	0.448	0.210	0
b(E2; 42 → 41)	0	0.0138	0.0624	0.167	0.313	0.339	0.311	0.271	0.273
b(E2; 61 → 41)	1.573	1.563	1.547	1.562	1.623	1.671	1.703	1.725	1.731
b(E2; 61 → 42)	0	10^{-3}	7.7×10^{-3}	0.035	0.052	0.033	0.011	0.0023	0

From the results of ref. [1]) and the data in table 3 it can be concluded that the reduced electric quadrupole transition probabilities between various rotational states of even nuclei can be divided into three types:

a) Transitions for which the reduced probabilities are of the order of

unity. Of this type are cascade transitions between levels of the ground rotational band and cascade transitions between "anomalous" rotational levels; examples are the transitions $3 \to 22$; $42 \to 3$; $42 \to 22$.

b) Transitions between levels of the ground rotational band and anomalous rotational levels of different spin. Examples are the transitions $3 \to 21$; $41 \to 22$, $42 \to 21$; $61 \to 42$. The reduced probabilities for such transitions are equal to zero for $\gamma = 0$ or $30°$ and are very small for other values of γ.

c) Transitions between levels of identical spin. Examples are $22 \to 21$, $42 \to 41$. The reduced probabilities for such transitions are equal to zero for $\gamma = 0$ and then significantly increase with increase of γ, attaining a maximum (of the order of unity) for $\gamma = 30°$. The $3 \to 41$ transition also belongs to this type of transition.

4. Quantum Number K and K Forbiddenness

The wave functions (2.3) for rotational states of even nuclei are represented by linear combinations † of the functions (2.4) corresponding to states with a definite value of the total nuclear angular momentum (quantum number K) projection on axis 3 in a coordinate system fixed to the nucleus. The magnitude of the coefficients determining the contribution of various terms to such a linear combination appreciably depends on the parameter γ which characterizes the deviation of the shape of the nucleus from axial symmetry. It can be seen from table 2 and ref. [1]) that for $\gamma < 15$ the wave functions of the nuclear rotational states can be approximated by expressions which contain only one value of K. Thus, the wave functions of spin 2 states can be replaced by the approximate functions

$$\begin{aligned} \psi^0_{21} &= |20\rangle = \left(\frac{5}{8\pi^2}\right)^{\frac{1}{2}} D^2_{m0}, \\ \psi^0_{22} &= |22\rangle = \left(\frac{5}{16\pi^2}\right)^{\frac{1}{2}} (D^2_{m2} + D^2_{m,-2}). \end{aligned} \tag{4.1}$$

The wave functions of spin 4 states can be approximated by the expressions

$$\begin{aligned} \psi^0_{41} &= |40\rangle = \left(\frac{9}{8\pi^2}\right)^{\frac{1}{2}} D^4_{m0}, \\ \psi^0_{42} &= |42\rangle = \left(\frac{9}{16\pi^2}\right)^{\frac{1}{2}} (D^4_{m2} + D^4_{m,-2}), \\ \psi^0_{43} &= |44\rangle = \left(\frac{9}{16\pi^2}\right)^{\frac{1}{2}} (D^4_{m4} + D^4_{m,-4}) \end{aligned} \tag{4.2}$$

etc.

† Only the function for a rotational state with a spin of 3 corresponds to a definite value of $K = 2$ (see ref. [1])).

In those cases ($\gamma < 15°$) when the rotational motion of the nucleus can be approximately described by wave functions of the (4.1) (4.2) type the rotational state of an even nucleus can be characterized by two quantum numbers, J and K. In this approximation values $K = 0$ correspond to levels of the "ground rotational band". Anomalous rotational levels can then be divided into a set of levels with $K = 2, 4, 6, \ldots$

The non-vanishing reduced probabilities (in our $e^2Q_0/16\pi$ units) for electric quadrupole transitions between states described by the approximate functions $|JK\rangle$ have the form

$$b_0(\mathrm{E2}; JK \to J'K) = 5\cos^2\gamma(2J0K|J'K)^2,$$
$$b_0(\mathrm{E2}; JK \to J'K+2) = \tfrac{5}{2}(1+\delta_{K0})\sin^2\gamma(2J2K|J'K+2)^2,$$
$$b_0(\mathrm{E2}; JK \to J'K-2) = \tfrac{5}{2}(1+\delta_{K2})\sin^2\gamma(2J-2K|J'K-2)^2.$$

The rules given at the end of the preceding section reduce in the approximation under consideration to the condition of validity of the equality

$$\Delta K = 0 \tag{4.3}$$

for the most probable transitions. Those transitions which occur when conditions (4.3) are not satisfied are usually called K-forbidden transitions. Since quantum number K is only approximate, rule (3.3) can be important only for nuclei with $\gamma < 15°$. If $\gamma > 15°$, the exact functions (2.3) should be employed. The results obtained in section 3 permit one to estimate the error which arises when the approximate functions $|JK\rangle$ are substituted for the exact functions (1.3).

It should be mentioned that in analyzing the data on the relative transition intensities a number of authors [3–6]) were compelled to assign a quantum number $K = 2$ to some of the levels possessing spins 3 and 2. Such excited states were ascribed by the authors to the so-called γ-vibrations, the frequency of the latter being chosen arbitrarily. However, if one views such states as rotational states of a non-axial nucleus a simple explanation of the spacing of the levels and of the relative transition probabilities between them can be proposed. In this case, only a single parameter γ which is uniquely defined by the ratio of the energies of two spin 2 levels is involved in the theory (see the following section).

5. Comparison with Experiment

In order to facilitate comparison of the results with the experimental data, plots of the dependence of parameter γ on the ratios of the rotational level energies for various spins are shown in fig. 1. The parameter γ can be uniquely determined from the ratio of the energies of two spin 2 levels with the aid of the formula

$$\frac{E_2(2)}{E_1(2)} = \frac{1+\sqrt{1-\frac{8}{9}\sin^2 3\gamma}}{1-\sqrt{1-\frac{8}{9}\sin^2 3\gamma}}. \tag{5.1}$$

After determining in this way the value of γ, one can use fig. 1 to find the position of the rotational levels of even nuclei.

It can be seen from fig. 1 that as a result of deviation of the shape of the nucleus from axial symmetry the interval rule 1 : 3.3 : 7 : 12 which is observed in the rotational band of axially symmetric nuclei is violated in the "ground rotational band". Thus, for example, for $\gamma = 30°$ the levels of the "ground rotational band" should satisfy the interval rule 1 : 2.67 : 5 : 8.

In fig. 1 the circles denote the experimental values of the ratio of the excitation energies of Os^{190}, Os^{192}, Dy^{160}, U^{232}, Pu^{238} [7]), U^{234} [8]), Os^{186}, Os^{188} [12]) and Er^{166}, Er^{168} [14]) nuclei to the energies of the first excited level of these nuclei. The figure shows that the theory yields spin sequences and energy ratios which are in good agreement with the experimental data. The slight difference between the experimental points and theoretical values can be removed by introducing a correction factor

$$-aJ^2(1+J)^2, \tag{5.2}$$

which accounts for the relation between rotation and internal excitation of the nucleus. The magnitude of this correction factor may serve as a criterion of validity of the adiabatic approximation. It should be noted that deviations of the experimental energy ratios from the interval rule 1 : 3.3 : 7 : 12 was wholly ascribed to violation of adiabaticity when the rotational spectra were analyzed from the viewpoint of axial symmetry of the nucleus. As a consequence the significance of the correction term (5.2) was overestimated. As a matter of fact deviation of axial nuclei from the interval rule is largely due to violation of axial symmetry of the nucleus. In this respect the experimental data on the 2^+, 4^+, 6^+ and 8^+ levels of the Os^{190} nucleus are especially interesting. The positions of these levels have been determined by Scharff-Goldhaber, Alburger, Harbottle and McKeown [9]) and Aten, de Feyfer, Sterk and Wapstra [10]) who studied cascade γ transitions in the decay of the 10 min Os^{190} isomer with a spin 10^-. The aforementioned authors [9, 10]) note that the experimental spin sequence satisfactorily agrees with the spin sequence of the levels in the Hf^{180} nucleus which possesses a pronounced rotational spectrum. However, the observed energy ratios were found to be appreciably different from the theoretical values for axially symmetric nuclei. It is also noted in the papers cited above that even the introduction of the correction (5.2) does not permit one to attain the experimental value 1 : 2.93 : 5.62 : 8.93.

Apart from a change in the interval rule for the levels of the ground rotational band, violation of axial symmetry of the nucleus leads to the appear-

ance of new rotational ("anomalous") levels. Such are, for example, the second levels with spins of 2 and 4 in the Os^{190} nucleus. These levels are not noticeable in decay of the 10 minute Os^{190} isomer but exhibit themselves in K-capture decay of Ir^{190}. The peculiarities of the spectrum of the Os^{190} nucleus are related to the large value of $\gamma = 21.4°$ which corresponds to the experimental energy ratio $E_2(2)/E_1(2) = 3.15$. With the value $\gamma = 21.4$ the reduced probabilities and relative electric quadrupole transition probabilities can be calculated for the Os^{190} nucleus. The values are given in table 4;

TABLE 4

Relative transition probabilities between various levels in the Os^{190} nucleus

E2 transitions	Transition energy (keV)	Reduced probabilities in $e^2Q_0^2/16\pi$ units	Relative probability
$21 \to 0$	186	0.934	1
$41 \to 21$	360	1.33	39.3
$22 \to 21$	400	0.467	21.5
$22 \to 0$	586	0.066	21.9
$22 \to 41$	40	0.071	3.5×10^{-5}
$3 \to 21$	604	0.18	69
$3 \to 22$	204	1.60	2.9
$3 \to 41$	244	0.515	2.14
$61 \to 41$	500	1.6	251
$42 \to 22$	540	0.461	100
$42 \to 41$	580	0.319	100
$42 \to 21$	940	0.007	24.6
$42 \to 3$	336	0.813	16.5
$42 \to 61$	80	0.06	9×10^{-4}

transitions referring to a theoretical level of spin 3 (which so far has not been observed experimentally) are also included. Using Os^{190} as an example we have demonstrated that the theory permits one to calculate the relative electric quadrupole transition probabilities between all rotational states.

The values for the relative transition intensities thus obtained yield a qualitative explanation of the observed decay schemes of the excited states of the Os^{190} nucleus. For example, decay of the isomer state (spin 10^-) of the Os^{190} nucleus occurs only as a result of a series of cascade transitions via rotational levels of the ground rotational band without noticeable excitation of anomalous rotational levels. The 4^+ and 2^+ anomalous rotational levels are excited in K capture of the Ir^{190} nucleus. From table 4 it follows that by emitting a E2 γ quantum a nucleus can go over from an excited state corresponding to the 4^+ anomalous rotational level to either a rotational state of spin 4 in the ground rotational band with subsequent cascade emission of two γ quanta or (with about the same probability) to a state 2^+ corresponding to the anomalous rotational band. From an anoma-

lous spin 2 level the transition can occur either to a state of spin 2 of the ground rotational band or (with about equal probability) directly to the ground state. According to the theory the Os^{190} nucleus should possess an excited state (about 790 keV) with a spin of 3. However, as can be seen from the table, excitation of this rotational state is not probable inasmuch as transition to other rotational states are much more probable from the (spin 4 and 6) levels located above it. This may explain why this rotational state hitherto has not yet been detected.

A comparison between the theoretical values of the transition probabilities and experimental values has been presented in ref. [1]). Since this paper was published, new experimental data on the ratio of the cascade ($22 \to 21$) and direct ($22 \to 0$) reduced probabilities have appeared. The experimental data of McGowan [11]) and Nathan [4]) and the theoretical values obtained on basis of the formula

$$\frac{b(\mathrm{E2};\, 22 \to 21)}{b(\mathrm{E2};\, 22 \to 0)} = \frac{20 \sin^2 3\gamma}{7\{9 - 8\sin^2 3\gamma - [3 - 2\sin^2 3\gamma]\sqrt{9 - 8\sin^2 3\gamma}\}}$$

are given in table 5.

TABLE 5

Ratio of reduced quadrupole transition probabilities for various nuclei

Nuclei	$\frac{E_2(2)}{E_1(2)}$	γ^0	$\frac{b(\mathrm{E2};\, 22 \to 21)}{b(\mathrm{E2};\, 22 \to 0)}$		$\frac{b(\mathrm{E2};\, 22 \to 21)}{b(\mathrm{E2};\, 21 \to 0)}$	
			theor	exp	theor	exp
Th^{232}	15.4	10.3	1.7	1.8	0.052	0.014
Gd^{154}	8.11	14.0	2.6	1.94	—	—
W^{184}	8.02	14.1	3.0	2.36	0.18	0.12
Os^{188}	4.08	19.3	4.5	2.7	0.32	0.19
Os^{190}	2.98	21.4	7.1	9.7	0.50	0.52
Os^{192}	2.37	25.4	21	9.1	0.90	0.68
Te^{122}	2.23	26.3	40	30	1.29	2.8
W^{186}	6.0	15.2	2.7	2.38	0.15	0.12

The parameter γ was computed from the ratio (5.1). The experimental data [11]) and theoretical values of

$$\frac{b(\mathrm{E2};\, 22 \to 21)}{b(\mathrm{E2};\, 21 \to 0)} = \frac{20 \sin^2 3\gamma}{7\{[9 - 8\sin^2 3\gamma + [3 - 2\sin^2(3\gamma)]\sqrt{9 - 8\sin^2 3\gamma}\}}$$

for the ratio of the probabilities of transitions ($22 \to 21$) and ($21 \to 0$) are given in the same table. It can be concluded from table 5 that the theory yields a unique relation between the indicated reduced transition probability ratios and ratio of the energies of both spin 2 levels.

Nathan and Waggoner [3]) have presented experimental values for the ratio of the intensities of the E2 transitions $3 \to 41$ and $3 \to 21$ in the Sm^{152} nucleus which imply that the experimental ratio of the reduced probabilities

$$\left\{\frac{b(E2;\, 3 \to 41)}{b(E2;\, 3 \to 21)}\right\}_{exp} = 1.88.$$

From the ratio $E_2(2)/E_1(2) = 8.9$ for the Sm^{152} nucleus it follows that $\gamma = 13.5°$ and the theory thus yields

$$\left\{\frac{b(E2;\, 3 \to 41)}{b(E2;\, 3 \to 21)}\right\}_{theor} = 1.37,$$

which is in good agreement with the experimental ratio.

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ON THE EXISTENCE OF SOLUTIONS OF THE BRUECKNER EQUATIONS FOR A MANY-FERMION SYSTEM

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Abstract: The Brueckner reaction matrix for a many-fermion system is frequently introduced as the formal "sum" of an infinite series which is a part of the complete many-body perturbation series.

It is shown that, when the system is infinite, for any "well-behaved" potential, there are always some unperturbed states for which the reaction matrix series diverges. Conditions for the existence of solutions of the integral equations for the reaction matrix are discussed and it is shown that "Cooper-type" singularities arise for potentials $v(r)$ with not too strongly bound states provided $\int_0^\infty v(r) \sin^2 k_F r dr$ is not positive ($\hbar k_F$ being the Fermi momentum). Explicit calculations show that there are other types of reaction matrix singularities for nuclear matter but not for liquid helium three. The results provide a potential foundation for Belyaev's calculations of nuclear spectra and suggest that there should be no energy gap in the spectrum of liquid helium three.

1. Introduction

The existence of "resonant" states in a many-fermion system was first demonstrated by Cooper [1]), who indicated that two particles near the Fermi surface could form a bound state if their mutual interaction were attractive. This observation provided the inspiration for the theory of superconductivity proposed by Bardeen, Cooper and Schrieffer [2]) in which, in particular, there appeared a gap in the energy spectrum of a superconductor, in agreement with experiment [3]). More recently, Goldstone [4]) and Gottfried [5]) have shown that, for certain interactions, the two-particle reaction matrix, which is the cornerstone of the Brueckner theory of a many fermion-system [6, 7]), could have singularities, and attempts have been made to link these divergences with an energy gap in the spectrum of the physical system under consideration [4, 5, 8]). Indeed, the operation of the exclusion principle provides a common origin for the existence of reaction matrix singularities and for the Cooper phenomenon.

At present the existence of singularities has been demonstrated for the special case of a separable potential or, at most, weak sufficiency conditions have been derived [4, 5]). The object of the present paper, therefore, is to obtain a more precise indication of the conditions which the two-body interaction must satisfy in order that singularities may occur and, in parti-

cular, to consider the special cases of physical interest-nuclei, superconductors and liquid helium three near absolute zero.

The methods to be used will also lead to a discussion of the convergence of the series expansion of the reaction matrix, which is of some importance, since the derivations of the Brueckner method which use perturbation theory [7]) introduce the reaction matrix by means of a rearrangement and partial summation of the complete perturbation series. Such a procedure would not be legitimate if the reaction matrix series were divergent and any conclusions drawn from these methods would have to be treated with caution.

In the next section, these considerations are illustrated by solving the equations for the special case of a δ-function interaction and the problem is restated in a form which is more suitable for generalization. Section 3 is devoted to a more general discussion of the equations and the convergence of the reaction matrix series is considered in section 4. Section 5 contains an application of the general methods to the situations of physical interest and the consequences of the results are discussed in section 6.

2. Illustration of the Problem

The Brueckner method [6, 7]) for calculating the energy of a system of fermions consists in replacing the potential energy operator v by the operator t defined by

$$t = v + v\frac{Q}{e}t; \tag{1}$$

Q and e will be defined more closely later. The matrix elements of t in any representation give the reaction matrix or t-matrix. The operator t is usually introduced as the sum of the series

$$t = v + v\frac{Q}{e}v + v\frac{Q}{e}v\frac{Q}{e}v + \ldots \tag{2}$$

which is a part of the complete many-body perturbation series in v.

Eq. (1) may be replaced by

$$\Omega = 1 + \frac{Q}{e}v\Omega \tag{3}$$

from which t is determined by

$$t = v\Omega. \tag{4}$$

The inter-particle potential v is assumed to represent a two-body force

which is velocity-independent except possibly for a spin-orbit term and an exchange character of the form of the nucleon-nucleon interaction used by Gammel and Thaler [9]).

For a very large system the unperturbed wave function is a Slater determinant of plane wave functions, spin functions and isobaric spin functions. Let $\mathbf{r}_i$ denote the position of particle i and let $\mathbf{k}$, $\mathbf{m}$ denote momenta respectively outside and inside the Fermi sphere whilst $\mathbf{l}$ represents either $\mathbf{k}$ or $\mathbf{m}$. Then, in mixed representation, eq. (3) becomes

$$\langle \mathbf{r}_i\mathbf{r}_j|\Omega|\mathbf{m}_i\mathbf{m}_j\rangle = \langle \mathbf{r}_i\mathbf{r}_j|\mathbf{m}_i\mathbf{m}_j\rangle$$
$$+ \int d\mathbf{r'}_i \int d\mathbf{r'}_j \int d\mathbf{k}_i \int d\mathbf{k}_j \frac{\langle \mathbf{r}_i\mathbf{r}_j|\mathbf{k}_i\mathbf{k}_j\rangle\langle \mathbf{k}_i\mathbf{k}_j|\mathbf{r'}_i\mathbf{r'}_j\rangle}{E(m_i)+E(m_j)-Ek_i)-E(k_j)} v(\mathbf{r'})\langle \mathbf{r'}_i\mathbf{r'}_j|\Omega|\mathbf{m}_i\mathbf{m}_j\rangle \quad (5)$$

where $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$ and

$$E(l_i) = \tfrac{1}{2}l_i^2 + V(l_i), \quad (6)$$

units being chosen so that $\hbar = 1$ and the mass of a particle is unity. $V(l_i)$ is the one-particle potential defined self-consistently by the relation

$$V(l_i) = \int d\mathbf{m}_i \langle \mathbf{l}_i\mathbf{m}_i|t|\mathbf{l}_i\mathbf{m}_i - \mathbf{m}_i\mathbf{l}_i\rangle. \quad (7)$$

The effect of the operator Q of eqs. (1), (2) and (3) is to limit the momentum space integrations in eq. (5) to the outside of the Fermi sphere.

In view of the self-consistency condition represented by eq. (7), it is not possible to discuss the existence of solutions of eq. (5) directly without presupposing a knowledge of these solutions. For this reason we put

$$E(l_i) = \tfrac{1}{2}l_i^2 \quad (8)$$

into the energy denominator of eq. (5), so that we now have an explicit equation to consider. The modifications caused by the inclusion of $V(l_i)$ will be discussed in section 5.

In order to write eq. (5) in a more manageable form we assume further that $v(r)$ is a central force and consider only those values of $\mathbf{m}_i$ and $\mathbf{m}_j$ which satisfy

$$\mathbf{m}_i + \mathbf{m}_j = 0.$$

Then, in s-state approximation, transforming to relative and centre of mass coordinates of particles i and j, eq. (5) takes the form

$$\psi(r) = \frac{\sin mr}{m} + \int_0^\infty dr' K(r, r')v(r')\psi(r') \quad (9)$$

where

$$K(r, r') = \frac{2}{\pi}\int_{k_F}^\infty dk \frac{\sin kr \sin kr'}{m^2 - k^2}, \quad (10)$$

m and **k** being relative momenta and k_F the Fermi momentum.

The effect of tensor forces and the extension to higher angular momentum states will be discussed in section 5.

By way of illustration let

$$v(r) = -\alpha\delta(r-a) \tag{11}$$

so that, for $r = a$, eq. (9) has the solution

$$\psi(a) = \frac{1}{m}\frac{\sin ma}{1+\alpha K(a, a)} \tag{12}$$

provided the equation

$$1+\alpha K(a, a) = 0 \tag{13}$$

is not satisfied. However, eq. (10) shows that $K(a, a) \to -\infty$ as $m \to k_F$ so that, unless either $\alpha < 0$, or $\alpha > 0$ together with

$$\frac{1}{\alpha} \leqq \frac{2}{\pi}\int_{k_F}^{\infty} dk \frac{\sin^2 ka}{k^2}, \tag{14}$$

there exists a value of m for which eq. (13) is satisfied and eq. (9) has no solution, i.e. the t-matrix has a singularity.

Moreover, the series of eq. (2) corresponds to the iteration solution of eq. (9) which, in this case, may be found by expanding the right hand side of eq. (12):

$$\psi(a) = \frac{\sin ma}{m}[1-\alpha K(a, a)+\alpha^2 K^2(a, a)- \ldots] \tag{15}$$

which is a convergent geometric series in $-\alpha K(a, a)$ provided

$$|\alpha K(a, a)| < 1. \tag{16}$$

However, since $K(a, a) \to -\infty$ as $m \to k_F$, there is, for any $\alpha \neq 0$, a value m_0 of m such that the series in eq. (15) diverges for all m satisfying $m_0 \leqq m \leqq k_F$ and converges otherwise. If $\alpha > 0$, i.e. $v(r)$ is attractive, the divergence is signalled by a singularity in $\psi(a)$; otherwise the integral equation has always a solution.

The source of the divergence may be traced to the cut-off in the lower limit of integration in eq. (10), since, if we put $k_F = 0$ therein and define the integral by taking the principal value at the singularity, eq. (12) becomes

$$\psi(a) = \frac{\sin ma}{m-\frac{1}{2}\alpha \sin 2ma}, \tag{17}$$

and provided

$$|\alpha a| < 1, \tag{18}$$

$\psi(a)$ has no singularity and the equivalent of the series of eq. (15) converges. Consequently, it seems reasonable to suppose that the occurrence of these divergences is not confined to potentials of the type assumed in eq. (11), and we now seek conditions under which a "well-behaved" potential $v(r)$ gives rise to similar difficulties.

A necessary condition that eq. (9) should have no solution is that the equation

$$\psi(r) = \int_0^\infty \mathrm{d}r' K(r, r') v(r') \psi(r') \tag{19}$$

should have a solution. This equation is a special case of the eigenvalue equation

$$\psi_i(r) = \lambda_i(m, k_\mathrm{F}) \int_0^\infty \mathrm{d}r' K(r, r') v(r') \psi_i(r') \tag{20}$$

and the problem of the existence of singularities is reduced to a discussion of the properties of the eigenvalues $\lambda_i(m, k_\mathrm{F})$ in order to find the conditions for which the equation

$$\lambda_i(m, k_\mathrm{F}) = 1 \tag{21}$$

is satisfied at the experimental value of k_F for some values of i and m. Moreover, when eq. (21) is satisfied, the series obtained by iterating eq. (9) is divergent. If we now introduce a coupling constant μ as a multiplier of $v(r)$, it follows that the potential $\mu v(r)$ gives rise to a singularity whenever

$$\mu = \lambda_i(m, k_\mathrm{F})$$

and that the iteration series of eq. (9), which is a power series in μ, diverges for this value of μ. Consequently, the radius of convergence of the series is equal to the modulus of the eigenvalue with lowest absolute value. (This result is precisely the well-known radius of convergence of the Neumann series [10]) solution of eq. (9).)

The series of eq. (2) will therefore diverge for a given potential $v(r)$ if there are values of i and m for which $|\lambda_i(m, k_\mathrm{F})| \leqq 1$.

3. Properties of the Eigenvalue Equation

In this section several properties of the eigenvalues $\lambda_i(m, k_\mathrm{F})$ of eq. (20) will be stated and proved. It is assumed the $v(r)$ represents a short-range force so that the integral in eq. (20) is essentially over a finite range, and the equation is not singular in virtue of an infinite range of integration. Writing $\langle r|k\rangle = \sin kr$, we assume that the matrix elements $\langle k|v|k'\rangle$ are finite. This, of course, would not be true for a potential which had an infinite hard core; however, the developments to be described in this section are valid if

the potential has an arbitrarily high but finite core. In order to apply the theory to nuclear matter, for example, in which the best available potentials have infinite repulsive cores [9]) we may either regard such potentials as limiting cases or argue that the experimental information does not at present exclude the possibility of a very high but finite core.

With these assumptions, the eigenvalues possess the following properties:

I. The eigenvalues are real and $\lambda_i(m, k_F) \neq 0$ unless $m = k_F$.

II. $\partial/\partial k_F|\lambda_i(m, k_F)| \geqq 0$ for $0 \leqq m < k_F$ and all i.

III. $\partial/\partial m|\lambda_i(m, k_F)| \leqq 0$ for $0 \leqq m < k_F$ and all i.

IV. (a) When $\langle k_F|v|k_F\rangle \neq 0$, $\lambda_i(m, k_F) \to 0$ as $m \to k_F$ for one and only one value n of i. If $\langle k_F|v|k_F\rangle < 0$, $\lambda_n(m, k_F) \to 0$ through positive values, otherwise $\lambda_n(m, k_F) \to 0$ through negative values.

(b) When $\langle k_F|v|k_F\rangle = 0$, $\lambda_i(m, k_F) \to 0$ as $m \to k_F$ for at least two values of i. In general there are exactly two such eigenvalues and one becomes zero from above and the other from below.

V. Let $k_F = 0 = m$. Then,

(a) if $v(r)$ has no bound state there is no eigenvalue in the range $0 < \lambda_i(0, 0) \leqq 1$;

(b) if $v(r)$ has N bound states there are N values of i for which $0 < \lambda_i(0, 0) \leqq 1$ and the equality holds only if at least one of the bound states has zero energy.

The remainder of this section is devoted to proving these properties.

From eq. (20) we find

$$\langle k'|v|\psi_i\rangle = \lambda_i(m, k_F)\,\frac{2}{\pi}\int_{k_F}^{\infty} \mathrm{d}k\,\frac{\langle k'|v|k\rangle\langle k|v|\psi_i\rangle}{m^2-k^2}$$

which may be written as an integral equation for $\langle k|v|\psi_i\rangle/|m^2-k^2|^{\frac{1}{2}}$ with a symmetric kernel and hence real eigenvalues [10]). Moreover, the trace of the kernel in eq. (20) is finite so that it may be inferred from the well-known Fredholm formulas [10]) that, for fixed values of k_F and m, the eigenvalues are discrete. This fact justifies the use of the subscript i (which takes on integral values) to label the eigenvalues and eigenfunctions.

Eq. (20) has a non-trivial solution when $\lambda_i(m, k_F) = 0$ only if $K(r, r')$ is infinite and $\lambda_i(m, k_F)K(r, r')$ is finite and non-zero. Eq. (10) shows that $K(r, r')$ is finite unless $m = k_F$ so that property I follows at once.

From eqs. (10) and (20),

$$\langle \psi_i|v|\psi_i\rangle = \lambda_i(m, k_F)\,\frac{2}{\pi}\int_{k_F}^{\infty} \mathrm{d}k\,\frac{|\langle \psi_i|v|k\rangle|^2}{m^2-k^2}. \tag{22}$$

The integrand on the right hand side of this equation is negative or zero everywhere (remembering that $m \leqq k_F$) so that

$$\langle \psi_i|v|\psi_i\rangle \neq 0 \tag{23}$$

unless either (a) $\lambda_i(m, k_F) = 0$
or (b) $\langle \psi_i|v|k\rangle = 0$ for all $k \geqq k_F$.

Condition (a) is satisfied only for $m = k_F$ and even then, as we shall see, the right hand side of eq. (22) is non-zero in general. Condition (b) implies that the right hand side of eq. (20) is zero unless $\lambda_i(m, k_F)$ is infinite and so for finite eigenvalues it is incompatible with a non-trivial solution of eq. (20).

If we now introduce the functional

$$J[\psi] = \frac{\frac{2}{\pi}\int_{k_F}^{\infty} \mathrm{d}k \frac{|\langle \psi|v|k\rangle|^2}{m^2-k^2}}{\langle \psi|v|\psi\rangle} \tag{24}$$

where $\langle \psi|v|\psi\rangle \neq 0$, it is easy to show that the extremals of $J[\psi]$ are the eigenfunctions of eq. (20) and that, for any extremal $|\psi_i\rangle$, the corresponding eigenvalue $\lambda_i(m, k_F)$ is given by

$$\frac{1}{\lambda_i(m, k_F)} = J[\psi_i]. \tag{25}$$

Moreover, since the integrand in eq. (24) is nowhere positive, it follows from eqs. (24) and (25) that $\lambda_i(m, k_F)$ and $\langle \psi_i|v|\psi_i\rangle$ have the opposite sign. Now, bearing in mind that $|\psi_i\rangle$ is an extremal of $J[\psi]$ and using eqs. (24) and (25), we see that

$$\frac{\partial}{\partial k_F}\lambda_i(m, k_F) = \frac{\lambda_i^2(m, k_F)}{\langle \psi_i|v|\psi_i\rangle} \frac{|\langle \psi_i|v|k_F\rangle|^2}{m^2-k_F^2} \tag{26}$$

so that $(\partial/\partial k_F)\lambda_i(m, k_F)$ has the opposite sign to $\langle \psi_i|v|\psi_i\rangle$ and so has the same sign as $\lambda_i(m, k_F)$. The second property listed above is therefore established and the third property may be proved in a similar manner.

Property IV may be derived from the Fredholm formulas [10]) but it is more convenient here to use the solution of eq. (20) for m in the neighbourhood of k_F.

Rewrite

$$\langle r|K|r'\rangle \equiv K(r, r') = L(m)[\sin k_F r \sin k_F r'] + G(r, r') \tag{27}$$

where

$$L(m) = \frac{1}{\pi m} \log\left(\frac{k_F-m}{k_F+m}\right), \tag{28}$$

$$\langle r|G|r'\rangle \equiv G(r, r') = \frac{2}{\pi}\int_{k_F}^{\infty} \mathrm{d}k \, \frac{\sin kr \sin kr' - \sin k_F r \sin k_F r'}{m^2-k^2}. \tag{29}$$

Thus $G(r, r')$ remains finite as $m \to k_F$ and the divergence is concentrated

into $L(m)$. Now for m near to k_F write the solution $|\psi\rangle$ of eq. (20) as

$$|\psi\rangle = |k_F\rangle + |\phi\rangle \tag{30}$$

and then, from eqs. (27), (28) and (29), we find, to first order

$$|\phi\rangle = \lambda G v |k_F\rangle, \tag{31}$$

$$\lambda L(m)\{\langle k_F|v|k_F\rangle + \lambda\langle k_F|vGv|k_F\rangle\} = 1, \tag{32}$$

λ being the eigenvalue corresponding to $|\psi\rangle$. Eq. (32) may be obtained as a first order approximation to the Fredholm series [10]) whose zeros give the eigenvalues. It can be seen that at least one root approaches zero as $m \to k_F$ and $L(m) \to \infty$, so that the first order approximation is consistent. Property IV is simply a restatement of the content of eq. (32).

Finally, it is necessary to consider eq. (20) when $m = k_F = 0$. In this case

$$K(r, r') = \begin{cases} -r & \text{for } r \leqq r' \\ -r' & \text{for } r > r' \end{cases} \tag{33}$$

and eq. (20) becomes

$$\psi_i(r) = -\lambda_i(0, 0)\int_0^r dr' r' v(r')\psi_i(r') - \lambda_i(0, 0) r \int_r^\infty dr' v(r')\psi_i(r'), \tag{34}$$

so that, since $v(r)$ is of short range,

$$\lim_{r\to\infty} \psi_i(r) = -\lambda_i(0, 0)\int_0^\infty dr' r' v(r')\psi_i(r'). \tag{35}$$

Moreover, differentiating both sides of eq. (34) twice with respect to r, we find

$$-\psi_i''(r) + \lambda_i(0, 0) v(r)\psi_i(r) = 0 \tag{36}$$

which is a zero energy Schrödinger equation whose solution for large r is

$$\psi_i(r) = c\left(1 - \frac{r}{a}\right), \tag{37}$$

where a is the scattering length. The eigenvalue condition is represented by eq. (35) which implies that a is infinite, i.e. there is a bound state of zero energy. Consequently, the eigenvalues $\lambda_i(0, 0)$ are the factors by which $v(r)$ must be multiplied in order that it may have a bound state of zero energy. The properties V(a) and V(b) may be inferred immediately.

4. Convergence of the Reaction Matrix Series

The third and fourth properties proved in section 3 show that for any $\varepsilon > 0$ and any value of $k_F \neq 0$ there exists a value m_0 of m such that

$$|\lambda_i(m, k_F)| < \varepsilon \tag{38}$$

for some value of i and all $m \geqq m_0$. Thus, according to the discussion at the end of section 2, the radius of convergence of the series expansion of t in powers of μ approaches zero as $m \to k_F$ and there is no potential satisfying the assumed conditions (short range and finite matrix elements $\langle k|v|k'\rangle$) for which this series converges for all values of m.

The divergence of the series of eq. (2) does not show that the complete perturbation series diverges but rather that the t-operator rearrangement of the complete series [7]) is illegitimate. A more satisfactory derivation of the Brueckner theory would not involve perturbation series and the t-operator would appear as the solution of the integral equation (1) rather than as the sum of the series of eq. (2) (see e.g. Brenig [11])).

The situation may be contrasted with the two-body probiem [12]), in which the integral equation occurs as the exact solution of the problem and the perturbation series, when it converges, merely represents a method of solving the integral equation; whereas, in the many-body problem, the integral equation is often introduced as a means of summing a series.

The divergence of the series expansion of t arises in such an extreme form only for very large systems in which the level spacing in intermediate states is very small. For finite systems the radius of convergence of the series in eq. (2) is non-zero and depends on the level spacing and on the details of the potential. At the same time it is clear that, for a given system, there is an upper limit to the strength of potential for which the series converges, so that the hard core of the nucleon-nucleon potential precludes convergence even for finite nuclei.

It should be stated that the definition of the t-operator given in eq. (1) may be no more than an inappropriate partial summation of the complete perturbation series and that inclusion of further terms would produce a convergent series. So far, however, no suitable redefinition of the t-operator has been proposed and the inclusion of "hole-hole" interactions, for example [13]), does not seem to resolve the difficulty.

Finally we remark that, although the single-particle potential of eqs. (6) and (7) has been omitted so far, the conclusions of this section are still valid provided this potential is continuous at the Fermi surface. If, however, it were possible to determine a self-consistent single-particle potential with an energy gap at the Fermi surface, the difficulty would be avoided for sufficiently weak potentials, but not for very strong ones.

5. Existence of Solutions of the Reaction Matrix Integral Equation

The problem now is to exhibit the conditions for which there exists an eigenvalue $\lambda_p(m, k_F)$ satisfying eq. (21) for some value m_0 of m at the observed value of k_F, since eq. (9) will then have no solution unless

$$\langle \psi_p | v | m_0 \rangle = 0. \tag{39}$$

This equation is not satisfied in general, although the special case of $m_0 = k_F$ and $p \neq n$ ($\lambda_n(m, k_F)$ being the particular eigenvalue of property IV) is an exception to this rule (see eqs. (30) and (41)). In virtue of property III of section 3, it is merely necessary to seek a value p of i for which $\lambda_p(0, k_F) \geqq 1$ whilst $\lambda_p(k_F, k_F) < 1$. In practice, $\lambda_i(0, k_F) \geqq 1$ for all i and observed values of k_F (see properties II and V) and no more than one $\lambda_i(k_F, k_F)$ is less than unity.

For any potential $\mu v(r)$, the reaction matrix will have a singularity when any of the eigenvalues $\lambda_i(m, k_F)$ is equal to μ and, for most values of i, a necessary condition for a singularity to arise from $\lambda_i(m, k_F)$ is $\mu > \mu_i(k_F)$, i.e. the potential has to be sufficiently strong. A similar statement holds good for the free-space reaction matrix, and the exclusion principle merely changes the critical values μ_i. However, there is one eigenvalue ($\lambda_n(m, k_F)$ which $\to 0$ as $m \to k_F$) for which the necessary condition for a singularity is $\mu \leqq \mu_0(k_F)$, i.e. the potential has to be sufficiently weak. (See properties IV and V.) This inversion is the principal consequence of the exclusion principle in determining the existence of reaction matrix singularities, and to emphasize this fact we shall refer to a singularity arising from the special eigenvalue $\lambda_n(m, k_F)$ as a "Cooper singularity"; in all other cases we shall refer to an "ordinary singularity".

The considerations of section 3 are most conveniently restated in terms of conditions that a given $v(r)$ could be seen to satisfy. It should be remembered that, at the moment, the energy denominator of eq. (5) contains kinetic energies only, so that the conditions stated below will be modified when a single particle potential is introduced (see the discussion later in this section).

Two broad divisions may be made:

A. $\langle k_F | v | k_F \rangle \leqq 0$

In this case a Cooper singularity may occur since $\lambda_n(m, k_F)$ of property IV is positive or zero. Properties II and V show that sufficient condition for the existence of a Cooper singularity is that $v(r)$ has no bound state or at most a bound state of zero energy and, in this case, there is a singularity for all values of k_F. If $v(r)$ has a bound state, then, according to properties II and V, there is a value $\bar{k}_F$ of k_F for which there is a singularity when $k_F \geqq \bar{k}_F$ and otherwise no singularity. The value of $\bar{k}_F$ has to be determined by a specific calculation and depends on the details of the potential. For a very strong potential it could happen that there is no Cooper singularity but there is an ordinary singularity. In these circumstances the reaction matrix elements could be positive everywhere so that the Brueckner method would not be useful.

The interaction which gives rise to superconductivity satisfies condition A (see refs. [1]) and [2]), so that we may expect a Cooper singularity to arise for superconductors.

B. $\langle k_F|v|k_F\rangle > 0$

In this case there is no Cooper singularity, since $\lambda_n(m, k_F)$ of property IV is negative everywhere.

If $v(r) \geqq 0$ everywhere (i.e. a purely repulsive potential) there is no ordinary singularity, since, by eqs. (24) and (25), all eigenvalues are negative. In all other circumstances, the details of $v(r)$ are important.

For cases of practical interest, $\lambda_i(0, k_F) \geqq 1$ for all i, and so it is desirable to find expressions for the $\lambda_i(k_F, k_F)$ in terms of phase shifts, since then the existence of singularities could be deduced from experimentally determined quantities. However, at present, no such expressions have been found and we must evaluate the eigenvalues by explicit calculations using assumed forms of $v(r)$ which have been inferred from phase shifts [9]), although it may be that the existence of a singularity depends on the details of the assumed form of potential.

The eigenvalues $\lambda_i(k_F, k_F)$ other than $\lambda_n(k_F, k_F) = 0$ may be found from the limit of the operator

$$F_m = K - \frac{|\psi_n\rangle\langle\psi_n|}{\langle\psi_n|v|\psi_n\rangle\lambda_n(m, k_F)} \tag{40}$$

as $m \to k_F$, K being defined in eq. (27).

It is easy to show that the eigenfunctions of eq. (20) satisfy the orthogonality condition

$$\langle\psi_i|v|\psi_n\rangle = 0 \tag{41}$$

when $\lambda_i(m, k_F) \neq \lambda_n(m, k_F)$. It follows then that, in this case, each eigenfunction of the operator $F_m v$ is an eigenfunction of Kv and that all eigenfunctions of Kv, other than $|\psi_n\rangle$, are given in this way. For m in the neighbourhood of k_F only one eigenvalue tends to zero and $F_m v$ tends to a finite limit Fv whose eigenvalues are the limits of the $\lambda_i(m, k_F)$ as $m \to k_F$. Using eqs. (30), (31) and (32) we find

$$F = \lim_{m \to k_F} F_m = G + \frac{\langle k_F|vGv|k_F\rangle}{|\langle k_F|v|k_F\rangle|^2}|k_F\rangle\langle k_F| - \frac{Gv|k_F\rangle\langle k_F|}{\langle k_F|v|k_F\rangle} - \frac{|k_F\rangle\langle k_F|vG}{\langle k_F|v|k_F\rangle}, \tag{42}$$

G being defined in eq. (29).

First we consider nuclear matter using the Gammel-Thaler potential whose spatial dependence is of the form

$$v(r) = \begin{cases} V_1 & \text{for } r < r_c \\ -V_0 \dfrac{e^{-\mu r}}{\mu r} & \text{for } r \geqq r_c; \end{cases} \tag{43}$$

V_1 is usually taken to be infinite, which would violate the assumption that $\langle k_F|v|k_F\rangle$ is finite. However, it makes no essential difference either to the interpretation of the existing scattering data or to the solution of the present problem if V_1 is taken to be very large and finite or, alternatively, infinite. Accordingly, to simplify the solution of the equations, we shall allow V_1 to become infinite.

We now use the method of Bethe and Goldstone and Brueckner and Gammel [14]) which is tantamount to making the replacement

$$v(r) \to \alpha\delta(r-r_c)+v(r) \qquad \text{for } r \geqq r_c \tag{44}$$

putting

$$v(r)\psi_i(r) = g_i(r) \qquad \text{for } r < r_c \tag{45}$$

and allowing $\alpha \to \infty$ and $\psi_i(r_c) \to 0$ so that $\alpha\psi_i(r_c) \to$ finite limit. If we further take a coordinate representation of eq. (42), and use the substitution (44), we find

$$\begin{aligned} \operatorname*{Lim}_{\alpha\to\infty}\langle r|F|r'\rangle \equiv F(r,r') = G(r,r')+G(r_c,r_c)\frac{\sin k_F r \sin k_F r'}{\sin^2 k_F r_c} \\ -\frac{\sin k_F r}{\sin k_F r_c}G(r_c,r')-\frac{\sin k_F r'}{\sin k_F r_c}G(r,r_c), \end{aligned} \tag{46}$$

$G(r,r')$ being defined in eq. (29). Now, neglecting $g_i(r)$ of eq. (45), (see ref.[14])) the equation to be solved becomes

$$\psi_i(r) = \lambda_i(k_F, k_F)\int_{r_c}^{\infty} F(r,r')v(r')\psi_i(r')\mathrm{d}r'. \tag{47}$$

This equation could have been obtained by putting the substitution (44) into eq. (20) and letting $m \to k_F$.

Eq. (47) was solved numerically using EDSAC II with $k_F = 1.5\times10^{13}$ cm^{-1} (the observed value) and using the singlet-even parameters ($r_c = 0.4 \times 10^{-13}$ cm, $V_0 = 434$ MeV, $\mu = 1.45\times10^{13}$ cm^{-1}) and the triplet even parameters ($r_c = 0.4\times10^{-13}$ cm, $V_0 = 877$ MeV, $\mu = 2.09\times10^{13}$ cm^{-1}) as given by Gammel and Thaler [9]).

The lowest eigenvalues $\lambda_i(k_F, k_F)$ were found to be 0.56 (singlet) and 0.75 (triplet) and in neither case was there another eigenvalue less than unity. Since the singlet potential has no bound state, all eigenvalues $\lambda_i(0, k_F)$ are greater than unity. For the triplet potential, the lowest of the $\lambda_i(0, k_F)$ was found to be 1.22. Consequently there is an ordinary singularity in both

cases although, of course, there is no Cooper singularity for nuclear matter.

In obtaining these results, tensor and spin-orbit forces have been neglected but it can be shown that they bring about a small decrease in the triplet eigenvalues and so do not affect the existence of a singularity. To discuss the effect of adding a single particle potential into the energy denominator, we assume that $V(l_i)$ is a monotonic increasing function of l_i. The denominator of the integrand in eq. (10) is then larger in absolute magnitude so that $|\lambda_i(m, k_F)|$ increases and singularities become less likely. It is therefore conceivable that one could find a single-particle potential which would remove the singularities. In the effective mass approximation, the eigenvalues are divided by the effective mass, M^*, so that a value of M^* less than 0.56 or 0.75 would be necessary to remove the singlet or triplet singularities respectively. In the calculations of Brueckner and Gammel [14]) which used the Gammel-Thaler potential, a self-consistent single particle potential was obtained with $M^* = 0.73$ at the Fermi surface. Since the effective mass increases as k increases, this potential will correspond to a "mean" effective mass larger than 0.73 and so the singlet singularity would not be removed and the triplet singularity is very unlikely to be removed. One must therefore conclude that the numerical approximations used by Brueckner and Gammel were not sufficiently accurate to show up the singularities.

At the moment, it does not seem to be worth while extending the calculations to odd-state potentials and to states of non-zero angular momentum and total momentum. The potentials are effectively weaker in odd-states and higher angular momentum states. The effect of the assumption of zero total momentum is obscure. However, the results of this section are unaffected by these considerations and the main conclusions of section 3 should not be altered in any essential way.

Finally we consider liquid helium three for which the potential of de Boer and Michels [15]) is given by

$$v(r) = \left(\frac{447}{r^{12}} - \frac{1.59}{r^6}\right) 10^{-12} \text{ erg},$$

where r is measured in Å.

This potential rises very steeply near 2.56 Å, so we approximate the short range behaviour by a hard core of this radius in order that the method used for the Gammel-Thaler potential might be applicable. At the experimental value of $k_F = 0.79$ Å^{-1}, it was found that the lowest eigenvalue was 7.1 so that, using property III, there is no singularity. The rather large value of the eigenvalue may be partially explained by the hard core approximation to the repulsive part of the potential [16]) but a more accurate treatment is unlikely to reduce the eigenvalue appreciably and the qualitative result that there is no singularity should certainly remain valid.

6. Discussion

We have seen that the existence of singularities in the Brueckner reaction matrix has a close connection with the Cooper phenomenon and that, although the formal reasons are not clear, it may have a corresponding connection with the existence of a superconducting state and a gap in the energy spectrum of the system under consideration. Recently, Bohr, Mottelson and Pines [17]) have demonstrated the evidence for an energy gap in nuclei and have suggested that a Cooper phenomenon is responsible. As a result of this proposal, Belyaev [18]) has used Bogolyubov's approach to the theory of superconductivity [19]) to calculate the spectra of nuclei between closed shells and the preliminary results appear to be reasonably successful. However, we have seen that nuclear matter has no Cooper singularity but rather an ordinary singularity, which may well be present for heavy nuclei but probably not for lighter nuclei. Therefore it seems to be likely that this second type of singularity rather than a Cooper phenomenon provides the foundation for Belyaev's work and, of course, the two effects are of a rather different physical nature.

In this connection, it is of interest to note that Soloviev [20]) using Bogolyubov's method [19]) has obtained conditions for the existence of "superfluidity" of nuclear matter which are similar to the inequality A of section 5.

Finally, although the present state of the theory reduces prediction to speculation, we may venture to say that the absence of a singularity for liquid helium three makes it unlikely that there is a gap in its energy spectrum.

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A NOTE ON THE INELASTIC SCATTERING OF HIGH ENERGY PROTONS BY LIGHT NUCLEI

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Abstract: The polarization for the inelastic scattering of 180 MeV protons by B^{11} is calculated using shell-model wave-functions in the L-S coupling extreme. Some discussion is given of the first 0^+-states in C^{12} and O^{16}.

In a previous paper [1]) (hereafter denoted by I) the differential cross-sections for the inelastic scattering to the 6.06 and 6.14 MeV levels of O^{16} and to the 4.5 MeV level of B^{11} were calculated using L-S coupled wave-functions. In the case of O^{16} the corresponding polarization has been reported earlier [2]) and the agreement with experiment was very good. For B^{11} the polarization is obtained from the equation [3])

$$P(\theta)\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2S\,\mathrm{tr}[T_{\mathrm{fi}}^{\dagger}\boldsymbol{\sigma}\cdot\mathbf{n}T_{\mathrm{fi}}] \tag{1}$$

were the quantities appearing in (1) are defined in I. Putting in the expression for $\mathrm{d}\sigma/\mathrm{d}\Omega$ leads after some simple algebra to the following result:

$$P(\theta) = \frac{2\,\mathrm{Re}\,\{(7a+\frac{1}{2}a_{\mathrm{pp}})^*(7c+\frac{1}{2}c_{\mathrm{pp}})+c_{\mathrm{pp}}{}^*b_{\mathrm{pp}}\}}{\{|7a+\frac{1}{2}a_{\mathrm{pp}}|^2+|7c+\frac{1}{2}c_{\mathrm{pp}}|^2+|\frac{1}{2}d_{\mathrm{pp}}|^2+|c_{\mathrm{pp}}|^2+|b_{\mathrm{pp}}|^2+|e_{\mathrm{pp}}|^2\}} \tag{2}$$

We can approximate this expression by a simpler one if we observe that the contribution from $a_{\mathrm{pp}}, \ldots, e_{\mathrm{pp}}$ is very small. Thus we have

$$P(\theta) = \frac{2\,\mathrm{Re}\,a^*c}{\{|a|^2+|c|^2\}}, \tag{3}$$

which is the well-known expression in the case of spin-zero nuclei with the excited state having $S = 0$ (see e.g. ref. [4])). The error introduced by this approximation is certainly less than 10 % within the interval 10°—30°. Fig. 1 shows the theoretical result and the experimental findings of A. Johansson *et al.* [5]), and the agreement is seen to be reasonable.

In I it was shown that by adding the differential cross-sections for the inelastic scattering of protons to the 6.06 and 6.14 MeV levels of O^{16} it is

possible to reproduce the experimental data fairly well. The 0^+-state was described by a wave-function identical to the ground-state wave-function except that one of the 1s-particles in the "alpha-core" was replaced by a 2s-particle. This model was successfully used by Redmond [6]) in connection

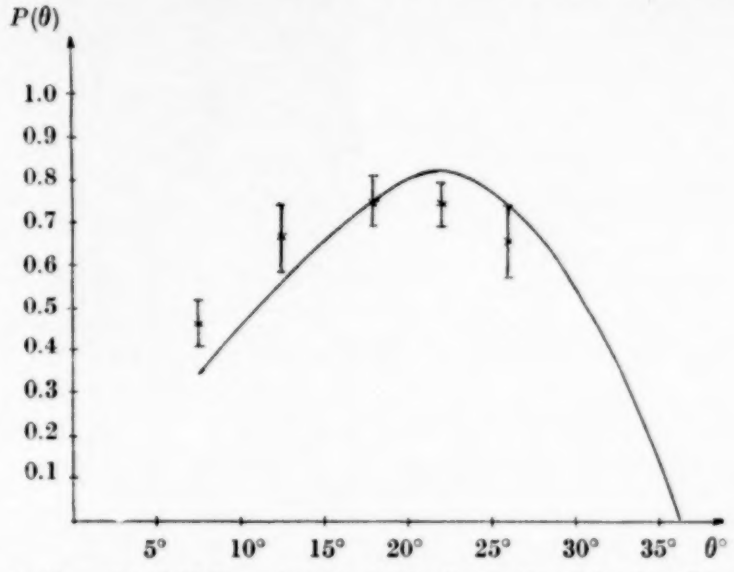

Fig. 1. Polarization of 180 MeV protons inelastically scattered by B^{11}. The points denote the experimental findings.

with electric monopole transitions in the nuclei C^{12} and O^{16}. It is interesting to see if this simple model is capable of accounting for the experimental results for C^{12}, because the 0^+-level of this nucleus is well separated from the near-lying levels. In the single-collision formalism used in I the absorptive part of the optical potential was assumed to be of Gaussian shape, and it is clear that the model is very sensitive to variations of the strength-parameter of this potential, since the incoming particle must penetrate the p-shell before reaching the "alpha-core". If we use a value of 60 MeV for this parameter [4]) the maximum value of the differential cross-section will be about 0.8 mb/sr. On the other hand, with the value 30 MeV (1) the corresponding maximum will be about 1.5 mb/sr. According to Tyrén *et al.* [7]) the experimental differential cross-section is very low and the maximum lies in the region between the theoretical values. However, it is very difficult to draw any definite conclusions regarding the validity of the model, depending on the uncertainty in using an optical potential for this light nucleus, and especially since it is not yet clear if it is possible to explain the low excitation energy of the first 0^+-states of C^{12} and O^{16} from the view-point of the above-mentioned model. In a paper by Ferrell and Visscher [8]), however, it is shown that for special linear combinations of excited shell-model configurations, involving 1s-excitation, the theoretical excitation energy for an individual nucleon is greatly reduced. Therefore it is reasonable to assume

that the model proposed by these authors, adjusted as regards the relative admixture of components of "Redmond-type", would satisfactorily account for the experimental data. Finally, observing that a small lowering of the calculated cross-sections can be obtained by a suitable inclusion of configurations of two-nucleon excitation into the wave-function, one can conclude that a model such as that proposed by Elliot[9]) would also be quite acceptable.

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THE PROPER FIELD OF A PHYSICAL NUCLEON

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Abstract: The energy values of Tomonaga's intermediate coupling equation for a physical nucleon are obtained from the asymptotic solutions of the equation when the number of mesons is very large. The energy values can be determined immediately from the requirement that the asymptotic solutions should contain only positive integer powers of meson emission operators. The possibility that the asymptotic solution for the ground state will give a good description of the situation at small distances from the core of nucleons is discussed. The result is consistent with the view that the production of meson showers by high energy nucleon collisions is due to the shaking off of meson clouds during collisions. No inconsistency with the low energy experiments is introduced by this picture of shower production.

1. Introduction

Recent discoveries of various types of strange particles seem to indicate that the structure of physical nucleons cannot be as simple as was thought before. Owing to the strong interactions of these particles with the nucleons, a physical nucleon would inevitably involve these particles in its virtual states. As a consequence the structure of the central portion of a physical nucleon would be greatly complicated. This may be the reason why relativistic treatments always lead to results which are not in agreement with experiment. With the further realization that the present form of quantum field theory may not even be adequate for the description of the central portion of physical nucleons, attempts have been made by Chew [1]) to replace the central portion by a fixed core and consider the outer portion as consisting only of π-mesons. The success of these attempts shows that virtual nucleon pairs and strange particles do not indeed interfere with the outer portion of the meson cloud.

In the treatment of Chew, a restriction on the number of mesons is made. This restriction is probably justified only in the outer fringe of the meson cloud. An alternative treatment not using this restriction is Tomonaga's intermediate coupling theory [2]). Friedman, Lee and Christian have shown [3]) that Tomonaga's theory leads to the same p-wave scattering of mesons by nucleons as obtained by Chew. It has further been demonstrated by Stroffolini [4]) that the meson scattering at a few hundred MeV is quite independent of the structure of the inner part of the meson cloud.

In the present paper we shall present a new method of determining the energy values of Tomonaga's equation without resorting to the usual variational calculus. It will be shown in section 2 that the asymptotic solutions of the equation can be obtained easily, and from them the energy values can be determined immediately from the condition that these solutions in Fock's representation should be series containing only positive integer powers of meson emission operators. Our treatment of Tomonaga's equation is similar to the strong coupling treatment of Pauli and Dancoff [5]) except that Fock's representation is used and that the strong-coupling condition is not needed in our case. The connection of our asymptotic solution with that obtained by the Bloch and Nordsieck approximation is also discussed.

In section 3 the possibility that our asymptotic solution may give a qualitative description of the situation at small distances from the core is discussed. The theory gives a satisfactory explanation of certain important features of the production of meson showers which cannot be explained by the statistic theory of Fermi [6]) or the hydrodynamical theory of Landau [7]).

2. Meson Field Outside the Core of a Physical Nucleon

We shall neglect the recoil of the core. The Hamiltonian operator for a nucleon at rest at the origin of the space coordinate system is given by

$$H = \sum_{\alpha=1}^{3} \sum_{\mathbf{k}} \left\{ \varepsilon \varphi_\alpha^*(\mathbf{k}) \varphi_\alpha(\mathbf{k}) + \frac{ig}{\sqrt{2\varepsilon}} U(k)\, \boldsymbol{\sigma} \cdot \mathbf{k}\, \tau_\alpha [\varphi_\alpha^*(\mathbf{k}) - \varphi_\alpha(\mathbf{k})] \right\} \tag{1}$$

where α represents the meson polarization in charge space, $\mathbf{k}$ the momentum vector of the meson, $\varepsilon = (\mu^2+\mathbf{k}^2)^{\frac{1}{2}}$, μ being the meson rest mass. Natural units $\hbar = c = \mu = 1$ are used so that $\varepsilon = (1+k^2)^{\frac{1}{2}}$. As usual, σ_i and τ_i $(i = 1, 2, 3)$ are respectively the spin and isobaric spin matrices of the nucleon; $\varphi_\alpha^*(\mathbf{k})$ and $\varphi_\alpha(\mathbf{k})$ are respectively emission and absorption operators for mesons of momentum $\mathbf{k}$ and charge α. They satisfy the following commutation relation

$$[\varphi_\alpha(\mathbf{k}), \varphi_\beta^*(\mathbf{k}')] = \delta_{\alpha\beta}\delta_{\mathbf{k}\mathbf{k}'}. \tag{2}$$

We may choose a set of real functions $F_n(k)$ satisfying the following orthonormal conditions:

$$\begin{gathered} \sum_{\mathbf{k}} k_i k_j F_n(k) F_m(k) = \delta_{ij}\delta_{nm} \\ (n, m = 1, 2, \ldots; \quad i, j = 1, 2, 3), \end{gathered} \tag{3}$$

and express $\varphi_\alpha(\mathbf{k})$ in the following form:

$$\varphi_\alpha(\mathbf{k}) = \sum_{i=1}^{3} \sum_{n=0}^{\infty} k_i F_n(k) \phi_{\alpha i}^{(n)} \tag{4}$$

where

$$\phi_{\alpha i}^{(n)} = \sum_{\mathbf{k}} F_n(k) k_i \varphi_\alpha(\mathbf{k}). \tag{5}$$

From (2), (4) and (5) we can easily verify that $\phi_{\alpha i}^{(n)}$ satisfies the following commutation relation:

$$[\phi_{\alpha i}^{(n)}, \phi_{\beta j}^{(m)*}] = \delta_{ij}\delta_{\alpha\beta}\delta_{nm}. \tag{6}$$

Inserting (4) into (1) gives

$$\begin{aligned} H = \sum_{i=1}^{3}\sum_{\alpha=1}^{3}\sum_{n=0}^{\infty} \Big\{ & A_n \phi_{\alpha i}^{(n)*}\phi_{\alpha i}^{(n)} + \sum_{m>n} B_{nm}[\phi_{\alpha i}^{(n)*}\phi_{\alpha i}^{(m)} + \phi_{\alpha i}^{(m)*}\phi_{\alpha i}^{(n)}] \\ & + \frac{ig}{\sqrt{2}} C_n \tau_\alpha \sigma_i [\phi_{\alpha i}^{(n)} - \phi_{\alpha i}^{(n)*}] \Big\} \end{aligned} \tag{7}$$

where

$$\begin{aligned} & A_n = \tfrac{1}{3}\sum_{\mathbf{k}} k^2 F_n(k)^2 \varepsilon, \quad B_{nm} = \tfrac{1}{3}\sum_{\mathbf{k}} k^2 F_n(k) F_m(k) \varepsilon, \\ & C_n = \tfrac{1}{3}\sum_{\mathbf{k}} \frac{U(k)}{\varepsilon^{\frac{1}{2}}} k^2 F_n(k). \end{aligned} \tag{8}$$

Sorting out terms containing $\phi_{\alpha i}^{(0)}$, we obtain

$$\begin{aligned} H = & \sum_{\alpha=1}^{3}\sum_{i=1}^{3} \Big\{ A_0 \phi_{\alpha i}^{(0)*}\phi_{\alpha i}^{(0)} + \frac{ig}{\sqrt{2}} C_0 \tau_\alpha \sigma_i [\phi_{\alpha i}^{(0)} - \phi_{\alpha i}^{(0)*}] \Big\} \\ & + \sum_{\alpha=1}^{3}\sum_{i=1}^{3}\sum_{n=1}^{\infty} \phi_{\alpha i}^{(n)*} \left[B_{0n}\phi_{\alpha i}^{(0)} - \frac{ig}{\sqrt{2}} C_n \tau_\alpha \sigma_i \right] \\ & + \sum_{\alpha=1}^{3}\sum_{i=1}^{3}\sum_{n=1}^{\infty} \phi_{\alpha i}^{(n)} \left[B_{0n}\phi_{\alpha i}^{(0)*} + \frac{ig}{\sqrt{2}} C_n \tau_\alpha \sigma_i \right] \\ & + \sum_{\alpha=1}^{3}\sum_{i=1}^{3}\sum_{n=1}^{\infty} \{ A_n \phi_{\alpha i}^{(n)*}\phi_{\alpha i}^{(n)} + \sum_{m>n} B_{nm}[\phi_{\alpha i}^{(n)*}\phi_{\alpha i}^{(m)} + \phi_{\alpha i}^{(m)*}\phi_{\alpha i}^{(n)}] \}. \end{aligned} \tag{9}$$

The Schrödinger equation of the present problem is

$$H\Psi = E\Psi. \tag{10}$$

There existed in the past two systematic approaches for solving the problem. One is the strong coupling approximation proposed by Wentzel [3]) and the other is the intermediate coupling approximation proposed by Tomonaga [2]). Both these authors made use of the single-wave-function approximation which in our notation means that Ψ should be a function of $\phi_{\alpha i}^{(0)*}$ only. In other words,

$$\phi_{\alpha i}^{(n)}\Psi \approx 0 \quad (n \neq 0). \tag{11}$$

From (9) and (11) we see that in Fock's representation, in which $\phi_{\alpha i}^{(n)}$ is replaced by $\partial/\partial\phi_{\alpha i}^{(n)*}$, Ψ must satisfy the following equations:

$$\sum_{\alpha=1}^{3}\sum_{i=1}^{3}\left\{A_0\phi_{\alpha i}^{(0)*}\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}+\frac{ig}{\sqrt{2}}C_0\tau_\alpha\sigma_i\left[\phi_{\alpha i}^{(0)*}-\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}\right]\right\}\Psi=E\Psi \quad (12)$$

and

$$\left\{B_{0n}\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}+\frac{ig}{\sqrt{2}}C_n\tau_\alpha\sigma_i\right\}\Psi\approx 0 \quad (n\neq 0). \quad (13)$$

Eq. (13) follows from the requirement that the emission operator $\phi_{\alpha i}^{(n)*}$ should be absent in (10). Tomonaga put

$$F_0(k)=\frac{U(k)}{\varepsilon^{\frac{3}{2}}I_1},\quad I_1=\tfrac{1}{3}\sum_{\mathbf{k}}\frac{U(k)k^2}{\varepsilon^3}. \quad (14)$$

It follows from (8) that

$$\begin{aligned}&C_0=I_1A_0,\quad C_n=B_{0n}I_1,\\ &A_0=\tfrac{1}{3}\sum_{\mathbf{k}}\frac{U(k)^2k^2}{\varepsilon^2I_1^{\,2}},\quad B_{0n}=\tfrac{1}{3}\sum_{\mathbf{k}}\frac{U(k)k^2}{\varepsilon^{\frac{1}{2}}I_1}F_n(k).\end{aligned} \quad (15)$$

Eqs. (12) and (13) then become

$$\sum_{\alpha=1}^{3}\sum_{i=1}^{3}\left\{\phi_{\alpha i}^{(0)*}\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}+\frac{ig}{\sqrt{2}}I_1\tau_\alpha\sigma_i\left[\phi_{\alpha i}^{(0)*}-\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}\right]\right\}\Psi=\lambda\Psi, \quad (16)$$

$$\lambda=E/A_0,$$

$$\left\{\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}+\frac{ig}{\sqrt{2}}I_1\tau_\alpha\sigma_i\right\}\Psi\approx 0. \quad (17)$$

Eq. (16) is just the equation obtained by Tomonaga using his intermediate-coupling approximation. The extent to which eq. (17) is satisfied may be used as a measure of the accuracy of his single-wave-function approximation.

It will be noted that the solution of (12) is independent of any particular choice of $F_0(k)$. The concrete choice of $F_0(k)$ is important only when we want to satisfy the condition (11). Therefore in the following we shall proceed to solve (12) without fixing the function $F_0(k)$.

In the present investigation we shall develop a new method to solve (9) and (10). First we notice that τ_α and σ_i are Pauli matrices with two columns and two rows. Following Pauli and Dancoff we may introduce orthogonal transformations in ordinary space and isobaric-spin space. Let τ'_r and σ'_i be components of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ after the transformations, we may write

$$\tau_\alpha=\sum_{r=1}^{3}B_{r\alpha}\tau'_r,\quad \sigma_i=\sum_{s=1}^{3}A_{is}\sigma'_s \quad (18)$$

where A and B satisfy the following conditions for orthogonal transformations:

$$\sum_{\alpha=1}^{3} B_{r\alpha}B_{s\alpha} = \delta_{rs}, \quad \sum_{r=1}^{3} B_{r\alpha}B_{r\beta} = \delta_{\alpha\beta};$$
$$\sum_{\alpha=1}^{3} A_{\alpha r}A_{\alpha s} = \delta_{rs}, \quad \sum_{r=1}^{3} A_{\alpha r}A_{\beta r} = \delta_{\alpha\beta}. \tag{19}$$

If we interpret the above transformations as rotations of coordinates as was done by Pauli and Dancoff, $A_{\alpha\beta}$ and B_{is} must be real numbers. In appendix 1 we shall show that if $A_{\alpha\beta}$ and B_{is} are allowed to be complex numbers the following results, valid when they are real numbers, still hold true.

We may choose A_{is} and $B_{r\alpha}$ satisfying the following condition:

$$\sum_{\alpha}\sum_{i} B_{r\alpha}\phi_{\alpha i}^{(0)*}A_{is} = \delta_{rs}Q_r. \tag{20}$$

Using (19) we can also write

$$\phi_{\alpha i}^{(0)*} = \sum_{r} A_{ir}B_{r\alpha}Q_r. \tag{21}$$

We notice that $\phi_{\alpha i}^{(0)}$ and $\phi_{\alpha i}^{(0)*}$ should be considered as a pair of complex conjugate numbers. In appendix 1 we have shown that A_{is} and $B_{r\alpha}$ which satisfy (20) and (21) must also be complex numbers. There we have also shown that $\pi_{\alpha i}^{(0)*}$ which is the dynamical variable conjugate to $\phi_{\alpha i}^{(0)*}$ is given by

$$\pi_{\alpha i}^{(0)*} = -i\frac{\partial}{\partial\phi_{\alpha i}^{(0)*}}$$
$$= \sum_{r}\sum_{k} A_{ir}B_{k\alpha}\left\{P_r\delta_{rk} + \tfrac{1}{2}\left[\frac{L^{rk}+T^{rk}}{Q_r-Q_k} - \frac{L^{rk}-T^{rk}}{Q_r+Q_k}\right](1-\delta_{rk})\right\} \tag{22}$$

where P_r is the momentum variable corresponding to Q_r, L^{rk} and T^{rk} are angular momentum operators in ordinary space and isobaric-spin space respectively. They satisfy the following commutation relations:

$$i[P_r, Q_r] = \delta_{rs}, \tag{23}$$
$$i[L^{rs}, A_{kt}] = \delta_{rt}A_{ks} - \delta_{st}A_{kr}, \tag{24}$$
$$i[T^{rs}, B_{t\alpha}] = \delta_{rt}B_{s\alpha} - \delta_{st}B_{r\alpha}. \tag{25}$$

The same expression (22) has been obtained by Pauli and Dancoff using the concept of rotations in ordinary and isobaric-spin spaces. It is now obtained by an algebraic method as in our case the transformation (18) cannot be interpreted directly as a rotation.

Inserting (21), (22), and (23) into (12), we obtain

$$\sum_{r}\left\{iA_0Q_rP_r + \frac{g}{\sqrt{2}}C_0\tau'_r\sigma'_rQ_r + i\frac{g}{\sqrt{2}}C_0\tau'_r\sigma'_rP_r + \sum_{k\neq r}\frac{g}{2\sqrt{2}}C_0\tau'_k\sigma'_rR_{rk}\right\}\Psi = E\Psi \tag{26}$$

where

$$R_{kr}=\frac{L^{kr}+T^{kr}}{Q_k-Q_r}-\frac{L^{kr}-T^{kr}}{Q_k+Q_r}. \tag{27}$$

We can easily verify the following relations:

$$[R_{12}, \tau'_1\sigma'_1Q_1+\tau'_2\sigma'_2Q_2]=2\tau_2\sigma_1, \qquad [R_{12}, \tau'_3\sigma'_3Q_3]=0 \tag{28}$$

and similar relations obtained by cyclic permutations of the indices. These relations will be useful in calculations reported in appendix 2.

It will be sufficient for our purpose to look for asymptotic solutions of (26) which hold when the number of mesons is very large. We shall show below that the eigenvalue E is determined already by the asymptotic solutions. After the asymptotic solutions are obtained, eq. (26) is needed only for continuing Ψ to the region where the number of mesons is small.

It is shown in appendix 2 that when the number of mesons is very large, the term containing R_{kr} in (26) can be neglected. (This can also be seen qualitatively from the fact that P_r and Q_r are both proportional to the square root of the number of mesons n: the term containing R_{kr} is therefore n times smaller than other terms in (26).) We obtain

$$\sum_r\left\{iQ_rP_r+\frac{ig}{\sqrt{2}}I\tau'_r\sigma'_rQ_r+\frac{g}{\sqrt{2}}I\tau'_r\sigma'_rP_r\right\}\Psi=\lambda\Psi, \qquad \lambda=E/A_0, \quad I=C_0/A_0. \tag{29}$$

We notice that the four quantities

$$1, \quad \tau'_1\sigma'_1, \quad \tau'_2\sigma'_2, \quad \tau'_3\sigma'_3$$

commute with one another. Consequently, although they are rather complicated matrices, they can still be handled as ordinary numbers in (29). Replacing iP_r by $\partial/\partial Q_r$, we obtain

$$\sum_{r=1}^{3}\left\{Q_r\frac{\partial}{\partial Q_r}+\frac{ig}{\sqrt{2}}I\tau'_r\sigma'_rQ_r-\frac{ig}{\sqrt{2}}I\tau'_r\sigma'_r\frac{\partial}{\partial Q_r}\right\}\Psi=\lambda\Psi. \tag{30}$$

Put

$$\Psi=\Psi_1(Q_1)\Psi_2(Q_2)\Psi_3(Q_3). \tag{31}$$

Eq. (30) becomes

$$\left\{Q_r\frac{\mathrm{d}}{\mathrm{d}Q_r}+\frac{ig}{\sqrt{2}}I\tau'_r\sigma'_rQ_r-\frac{ig}{\sqrt{2}}\tau'_r\sigma'_r\frac{\mathrm{d}}{\mathrm{d}Q_r}\right\}\Psi_r=\lambda_r\Psi_r, \quad (r=1, 2, 3), \tag{32}$$

and

$$\lambda=\lambda_1+\lambda_2+\lambda_3. \tag{33}$$

Eqs. (32) are three total differential equations of the first order, the general solution of which is found immediately to be

$$\Psi_r = A_r\left[Q_r - \frac{ig}{\sqrt{2}} I\tau'_r\sigma'_r\right]^{(\frac{1}{4}g^2I^2+\lambda_r)} \exp\left\{-\frac{ig}{\sqrt{2}} I\tau'_r\sigma'_r Q_r\right\} \tag{34}$$

where A_r is the normalization constant. By definition Ψ_r must be a single valued power series of Q_r; this requires that

$$n_r = \tfrac{1}{4}g^2I^2+\lambda_r \tag{35}$$

must be an integer. It will easily be seen that n_r cannot be a negative integer. If n_r were negative, Ψ_r would tend to infinity when $Q_r \to -\frac{1}{2}gI\tau'_r\sigma'_r$. Since such an infinite value cannot have any physical meaning, we conclude that n_r can only be a positive integer. From (35) we obtain

$$E = A_0(\lambda_1+\lambda_2+\lambda_3) = A_0(n_1+n_2+n_3-\tfrac{3}{4}g^2I^2). \tag{36}$$

The ground state of the system is given by $n_r = 0 (r = 1, 2, 3)$. The solution of (32) for the ground state is given by

$$\begin{aligned} \Psi_0 &= A \exp\left\{-\frac{ig}{\sqrt{2}} I \sum_{r=1}^{3} \tau'_r\sigma'_r Q_r\right\}, \\ E_0 &= A_0\lambda_0 = -\tfrac{3}{4}g^2I^2A_0. \end{aligned} \tag{37}$$

Since

$$\sum_{r=1}^{3} \tau'_r\sigma'_r Q_r = \sum_{\alpha=1}^{3}\sum_{i=1}^{3} \tau_\alpha\sigma_i\phi_{\alpha i}^{(0)*} \tag{38}$$

eq. (37) can also be written as

$$\Psi_0 = A \exp\left\{-\frac{ig}{\sqrt{2}} I \sum_{\alpha}\sum_{i} \tau_\alpha\sigma_i\phi_{\alpha i}^{(0)*}\right\}. \tag{39}$$

Using (5) and (8) we obtain further

$$\Psi_0 = A \exp\left\{-\frac{ig}{\sqrt{2}} \frac{\sum_{\mathbf{k}} U(k)\varepsilon^{-\frac{1}{2}}k^2F_0(k)}{\sum_{\mathbf{k}} k^2\varepsilon^{-1}F_0(k)^2} \sum_{\alpha}\sum_{\mathbf{k}} \tau_\alpha(\boldsymbol{\sigma}\cdot\mathbf{k})F_0(k)\varphi_\alpha^*(\mathbf{k})\right\}. \tag{40}$$

Inserting Tomonaga's assumption for $F(k)$ given by (14) into (40), we obtain

$$\Psi_0 = A_1 \exp\left\{-\frac{ig}{\sqrt{2}} \sum_{\alpha}\sum_{\mathbf{k}} \tau_\alpha(\boldsymbol{\sigma}\cdot\mathbf{k}) \frac{U(k)}{\varepsilon^{\frac{1}{2}}} \varphi_\alpha^*(\mathbf{k})\right\}. \tag{41}$$

It will be pointed out that Tomonaga's expression for $F_0(k)$, namely (14), does not satisfy (11) very well. To see this we insert (21) and (22) into eq. (13) which is equivalent to (11), and obtain

$$B_{0n}\left\{iP_r + \frac{g}{\sqrt{2}} I_1^{\frac{1}{2}}\tau'_r\sigma'_r\right\}\Psi \approx 0 \qquad (\text{when } r = s), \tag{43}$$

$$B_{0n}\left\{i\tfrac{1}{2}R_{rs} + \frac{g}{\sqrt{2}} I_1^{\frac{1}{2}}\tau'_s\sigma'_r\right\}\Psi \approx 0 \quad (\text{when } r \neq s). \tag{44}$$

It can easily be verified by inserting (41) that the left-hand side of (43) vanishes identically, but the left-hand side of (44) is different from zero. It is difficult to assert from this how far this would violate the one-wave-function assumption. To find a better $F_0(k)$ various authors have resorted to variational methods. For our calculation in the following section it is sufficient to leave $F_0(k)$ indeterminate. Eq. (40) can also be written in the following form:

$$\Psi_0 = A \exp\left\{-\frac{ig'}{\sqrt{2}} \sum_{\alpha} \sum_{\mathbf{k}} \tau_\alpha (\boldsymbol{\sigma} \cdot \mathbf{k}) F_0(k) \varphi_\alpha^*(k)\right\} \tag{45}$$

where

$$g' = g \sum_{\mathbf{k}} U(k) \varepsilon^{-\frac{1}{2}} k^2 F_0(k) / \sum_{\mathbf{k}} k^2 F_0(k)^2 \varepsilon. \tag{46}$$

It will be noted that (41) is just the solution of (1) if from the beginning we consider τ_α and σ_i as two classical unit vectors. An approximation of a similar nature has been used before by Oppenheimer, Lewis and Wouthuysen [9]) in treating the problem of multiple production of mesons by nucleon-nucleon collisions. It may be considered as the generalization of the Bloch-Nordsieck approximation [10]) in quantum electrodynamics, in which the Dirac matrix α_i is also treated as a classical unit vector. This approximation has also been used by the present author [11]) several years ago to obtain the same solution (41).

3. The Meson Field at small Distances from the Core

In the foregoing section the asymptotic solutions are used only as a mathematical aid for the determination of energy values of Tomonaga's equation (16). Our result will be perfectly correct even though states with large number of mesons turn out eventually to be unimportant in comparison with those with small numbers of mesons. In the present section we shall investigate whether the asymptotic solution for the ground state can give a qualitative description of the situation at small distances from the core, as one would expect [12]) the interaction to be rather strong in this region and therefore the number of mesons there should be the largest. To begin with, it is perhaps worthwhile to point out that experiments have so far revealed very little about the structure of the inner part of the meson cloud of a nucleon. As has already been stressed by Fubini and Thirring [13]), no definite statements about the magnetic moments, electromagnetic radii and photoproduction of mesons can be made before the electromagnetic properties of the core are known. The only experimental material which can be used for a severe comparison with the theory is the p-wave meson-nucleon scattering. However, as has been shown by Stroffolini [4]), even this experiment cannot lead to definite statements about the structure of the inner part of the meson

cloud. The nuclear potential cannot give any indication about the number of mesons in the field since it is well known that in neutral scalar theory the Yukawa potential is always obtained independent of the number of virtual mesons in the field; the same situation also occurs in the strong coupling limit for all types of meson fields.

From the asymptotic solution (45) for large n the probability P_n of finding n mesons in the proper field of a nucleon can be obtained immediately as

$$P_n = \frac{1}{n!}|A|^2 \frac{g'^{2n}}{2^n}\{\sum_{\mathbf{k}} F_0(k)^2 k^2\}^n. \tag{47}$$

Replacing summation by integration, we obtain

$$P_n = \frac{1}{n!}|A|^2 \frac{g'^{2n}}{2^n}\left\{\frac{1}{(2\pi)^3}\int_0^\infty F_0(k) k^2 \, 4\pi k^2 \mathrm{d}k\right\}^n. \tag{48}$$

Eq. (48) gives the distribution of the number of mesons outside the core. It should be remembered that n in (48) should be sufficiently large to warrant the use of the asymptotic solution (45). When n is very large, the quantity (48) will decrease monotonically as n increases. An interesting situation may arise in which P_n first increases with n and then after passing through a maximum decreases monotonically. From (48) we find that this maximum occurs at $n = N$ where

$$N = \tfrac{1}{2}g'^2 \frac{1}{2\pi^2}\int_0^\infty F_2(k)^2 k^4 \mathrm{d}k. \tag{49}$$

As we have mentioned before, N must be sufficiently large to ensure that the maximum derived from the asymptotic solution represents a real maximum. This N will also give a measure of the average number of mesons in the field. In the following we shall see that this will be the case if the intrinsic interaction constant is sufficiently large.

Eq. (49) shows that the most probable number of mesons N is determined by the function $F_0(k)$ and the coupling constant g'. In the last section we have considered Tomonaga's expression for $F_0(k)$:

$$F_0(k) = \frac{U(k)}{\varepsilon^{\frac{1}{2}} I_1}, \quad I_1^2 = \tfrac{1}{3}\frac{1}{(2\pi)^3}\int_0^\infty \frac{U(k)^2 k^2}{\varepsilon^3} 4\pi k^2 \, \mathrm{d}k. \tag{50a}$$

Using a variation method Friedman, Lee and Christian [3]) obtained the following expression:

$$F_0(k) = \frac{U(k)}{\varepsilon^{\frac{1}{2}}(\varepsilon+a) I_2}, \quad I_2^2 = \tfrac{1}{3}\frac{1}{(2\pi)^3}\int_0^\infty \frac{U(k)^2 k^2}{\varepsilon(\varepsilon+a)^2} 4\pi k^2 \mathrm{d}k \tag{50b}$$

where a is a constant. When a is very large, (50b) becomes

$$F_0(k) = \frac{U(k)}{\varepsilon^{\frac{1}{2}} I_3}, \quad I_3^2 = \tfrac{1}{3}\frac{1}{(2\pi)^3}\int_0^\infty \frac{U(k)^2 k^2}{\varepsilon} 4\pi k^2 \, \mathrm{d}k. \tag{50c}$$

From the result of the last section we have

$$g'^2 = g\int_0^\infty U(k)F_0(k)k^4\varepsilon^{-\frac{1}{2}}\mathrm{d}k/\int_0^\infty F_0(k)^2k^4\mathrm{d}k.$$

In the following we shall estimate the value of N using Tomonaga's expression (50a) and also the extreme case (50c). The expression (50b) of Friedman *et al.* must lead to a result intermediate between the above two cases. The following three types of cut-off function $U(k)$ will be used:

(i) straight cut-off $U(k) = 1$ for $k \leqq K$, $= 0$ for $k > K$;

(ii) Gaussian cut-off $U(k) = \mathrm{e}^{-k^2/2K^2}$;

(iii) exponential cut-off $U(k) = \mathrm{e}^{-k/K}$.

For K we may use the inverse Compton wave length of the nucleon: $K \approx 7$.

Before we proceed to calculate N we must also know the intrinsic interaction constant g. This g is different from the observed or renormalized constant g_r by a renormalization factor R, i.e., $g^2 = g_\mathrm{r}^2 R$. According to the calculation of Friedman *et al.*, R is found to be as large as 7. On the other hand, a recent estimation by Fubini and Thirring [13]) gives a much smaller value 2.2. The discrepancy between these two values can be explained as follows: As pointed out by Stroffolini [4]), the solution of Friedman *et al.* describes more accurately the situation at small distances from the core where the energy of the meson field is large, while the solution of Fubini and Thirring, which correlates the experimental results in the low energy region, describes more accurately the situation in the outer region of the meson cloud. Consequently, the effect of charge renormalization is likely to be underestimated in the calculation of Fubini and Thirring owing to the neglect of contributions from high energy mesons at small distances from the core. We think that the value obtained by Friedman *et al.* is a better one.

That the renormalization factor could be quite large can be seen from the following consideration. It should be noted that in many fixed source theories the renormalization factor R is finite. This factor is equal to the ratio of the probability for the bare nucleon to emit or absorb a meson of given charge to the same probability for the physical nucleon. Thus, for the neutral scalar theory this renormalization factor is 1, since the core of the nucleon does not lose any ability to emit or absorb a meson owing to the presence of the meson cloud. For the charged scalar theory the upper limit of this factor is 2, which corresponds to the strong coupling limit when the core has equal probabilities of being a bare proton and a bare neutron. By similar reasons, for the charge-symmetrical scalar theory the strong coupling limit gives the value 3. In our case of charge-symmetrical pseudo-scalar theory, it can easily be seen that the upper limit for the renormalization factor is 9 (the isobaric-spin and the ordinary spaces each contribute a factor 3). The value 7 obtained by Friedman *et al.* corresponds to the situation that the orientation of the core is 80 % at random in the isobaric and ordinary spin spaces.

It will further be pointed out that the largeness of R is also a necessary condition for our calculation to be self-consistent. In deriving (49) we have required that N should be sufficiently large. It can easily be seen that when N is large, the orientation of the core in isobaric and ordinary spin spaces must be quite random. From the above consideration it follows that R cannot be much less than the limiting value 9.

If we accept the value $R = 7$ and use the renormalized coupling constant determined from experiment $g_r^2/4\pi = 0.1$, then inserting (50)—(52) into (49), we can calculate the numerical values of N for different choices of $F_0(k)$ and $U(k)$. The results are presented in table 1 †. We see from this

TABLE 1

$F(k)$	$U(k)/\varepsilon^{\frac{1}{2}} I_3$			$U(k)/\varepsilon^{\frac{1}{2}} I_1$		
$U(k)$	straight cut-off	Gaussian cut-off	expon. cut-off	straight cut-off	Gaussian cut-off	expon. cut-off
N	4.6	3.2	4.2	5	5	13

table that it is possible for N to be quite large. There is an argument by Cini and Fubini [14]) for favouring the exponential cut-off function which has a tail in the high energy region. The value N depends rather sensitively on the cut-off momentum K and the renormalization factor R. It should also be noted that for the distribution (41) the fluctuation of the number of mesons around the mean value N is quite large, so that even when N is not large, the probability of finding large numbers of mesons is still appreciable. From the result of appendix 2 we see that $N = 5$ would be sufficient for justifying the use of asymptotic solution.

Based on the above picture of the meson cloud at small distances from the core, The "S-star" observed by Lord, Fainberg and Schein [15]) may be explained immediately as formed by the shaking off of the meson clouds of two colliding nucleons. In the centre of mass system the shower particles shaken off from each nucleon form a narrow cone moving in the direction of the initial nucleon.

From the above picture we also see that the average transverse component of momentum of the shower particles perpendicular to the line of collision should be $\approx K$ and independent of the energy of collision. This is also in agreement with experiment. We therefore conclude that the phenomena of meson shower production by nucleon collisions at energy 10^{17} eV fit quite well into the above picture of the structure of the meson cloud of nucleons. To explain the phenomena there is no urgent need to go to Fermi's statistical theory or the hydrodynamical and non-linear theories. One decisive defect

† In calculating N we have replaced ε by k in all integrals since the contribution to these integrals comes mainly from large values of k.

of all these theories is that they cannot give a satisfactory explanation why anti-nucleons and strange particles are not produced in large quantities alongside with the shower of mesons.

Appendix 1

A large part of the following results has been obtained by Pauli and Dancoff before. But in their calculations the meaning of $\phi_{\alpha s}^{(0)}$ is entirely different. In their case $\phi_{\alpha s}^{(0)}$, Q_r are all real quantities, whereas our $\phi_{\alpha s}^{(0)*}$ and Q_r must be considered as complex quantities. Therefore, in our calculations we shall avoid to use the geometric meaning of A_{sk} and $B_{r\alpha}$ and their representation in terms of Euler angles.

We introduce the following transformations in the ordinary and isobaric spin spaces:

$$\sigma_s = \sum_{k=1}^{3} A_{sk}\sigma'_k, \tag{1'}$$

$$\tau_\alpha = \sum_{r=1}^{3} B_{r\alpha}\sigma'_r; \tag{2'}$$

A_{sk} and $B_{r\alpha}$ are all complex numbers but satisfy the following conditions of orthogonal transformation:

$$\begin{aligned} &\sum_{\alpha} B_{r\alpha}B_{s\alpha} = \delta_{rs}, \quad \sum_{r} B_{r\alpha}B_{r\beta} = \delta_{\alpha\beta}, \\ &\sum_{\alpha} A_{\alpha r}A_{\alpha s} = \delta_{rs}, \quad \sum_{r} A_{\alpha r}A_{\beta r} = \delta_{\alpha\beta}. \end{aligned} \tag{3'}$$

We may choose A_{ks} and $B_{r\alpha}$ such that

$$\sum_{\alpha, s} B_{r\alpha}\varphi_{\alpha s}A_{sk} = \delta_{rk}Q_r \qquad (\varphi_{\alpha s} \equiv \phi_{\alpha s}^{(0)*}). \tag{4'}$$

In order to prove that this is possible, we consider the following two matrices:

$$C_{rs} = C_{sr} = \sum_{\alpha} \varphi_{\alpha r}\varphi_{\alpha s}, \quad C'_{\alpha\beta} = C'_{\beta\alpha} = \sum_{r} \varphi_{\alpha r}\varphi_{\beta r}; \tag{5'}$$

C_{rs} and $C'_{\alpha\beta}$ should be considered as complex numbers.

Since any symmetrical complex matrix can be transformed into a diagonal matrix by a complex orthogonal transformation, we may choose A_{sk} and $B_{r\alpha}$ to satisfy

$$\sum_{r, s} A_{r\alpha}C_{rs}A_{s\beta} = \delta_{\alpha\beta}W_\alpha, \tag{6'}$$

$$\sum_{\alpha, \beta} B_{r\alpha}C'_{\alpha\beta}B_{s\beta} = \delta_{rs}W'_r. \tag{7'}$$

Using (5') and (3'), the above two relations can be written

$$\sum_{t} \Omega_{t\alpha}\Omega_{t\beta} = \delta_{\alpha\beta}W_\alpha, \tag{8'}$$

$$\sum_{t} \Omega_{st}\Omega_{rt} = \delta_{rs}W'_r, \tag{9'}$$

where

$$\Omega_{t\beta} = \sum_{r,s} B_{tr}\varphi_{rs}A_{s\beta}. \tag{10'}$$

Eq. (6') and (7') can also be written as

$$\sum_{s} C_{rs}A_{s\beta} = A_{r\beta}W_{\beta}, \tag{11'}$$

$$\sum_{r} B_{r\alpha}C_{\alpha\beta} = B_{r\beta}W'_{r}. \tag{12'}$$

Since

$$\sum_{r,\alpha} \varphi_{\rho r}\varphi_{\alpha r}\varphi_{\alpha s} = \sum_{r} \varphi_{\rho r}C_{rs} = \sum_{\alpha} C'_{\rho\alpha}\varphi_{\alpha s}, \tag{13'}$$

multiplying (11') from the right by $\varphi_{\rho r}$, we obtain

$$\sum_{r,s} \varphi_{\rho r}C_{rs}A_{s\beta} = \sum_{\alpha,s} C'_{\rho\alpha}\varphi_{\alpha s}A_{s\beta} = \sum_{r} \varphi_{\rho r}A_{r\beta}W_{\beta}.$$

Multiply again by $B_{n\rho}$;

$$\sum_{\alpha,s,\rho} B_{n\rho}C'_{\rho\alpha}\varphi_{\alpha s}A_{s\beta} = \sum_{\rho,r} B_{n\rho}\varphi_{\rho r}A_{r\beta}W_{\beta}.$$

From (12') we obtain

$$\sum_{\alpha,s} B_{n\alpha}W'_{n}\varphi_{\alpha s}A_{s\beta} = \sum_{r,\rho} B_{n\rho}\varphi_{\rho r}A_{r\beta}W_{\beta}$$

or

$$W'_{n}\Omega_{n\beta} = \Omega_{n\beta}W_{\beta}. \tag{14'}$$

If W'_1, W'_2 and W'_3 are all distinct, (14') gives immediately

$$W'_{n} = W_{n}, \tag{15'}$$

and shows that $\Omega_{\alpha\beta}$ is a diagonal matrix. If W'_1, W'_2 and W'_3 are not distinct, e.g., $W'_1 = W'_2$, we may consider the case as the limiting case of $W'_1 \to W'_2$ and obtain the same result. From (8) and (9) we have

$$\Omega_{r\alpha} = \sum_{r,s} B_{rr}\varphi_{rs}A_{s\alpha} = \pm\delta_{r\alpha}\sqrt{W_r}. \tag{16'}$$

Eq. (16) proves that (4') is possible and $Q_r = \pm\sqrt{W_r}$.

In the following we shall use (3') to derive some important relations:

(i) The angular momentum operator of the meson field is

$$L_{ik} = \sum_{r} (\varphi_{\alpha i}\pi_{\alpha k} - \varphi_{\alpha k}\pi_{\alpha i}) \qquad (\pi_{\alpha i} \equiv \pi^{(0)*}_{\alpha i}). \tag{17'}$$

Using commutation relations between φ_α and φ_β we prove easily

$$i[L_{ik}, \sum_{s} \varphi_{\alpha s}\varphi_{\beta s}] = 0, \tag{18'}$$

$$i[L_{ik}, \sum_{\alpha} \varphi_{\alpha s}\varphi_{\alpha s}] = 2\sum_{\alpha} \varphi_{\alpha i}\varphi_{\alpha s}\delta_{ks} - 2\sum_{\alpha} \varphi_{\alpha k}\varphi_{\alpha s}\delta_{is}. \tag{19'}$$

these two relations can also be written as

$$i[L_{ik}, C'_{\alpha\beta}] = 0, \tag{20'}$$

$$i[L_{ik}, C_{ss}] = 2C_{is}\delta_{ks} - 2C_{ks}\delta_{is}. \tag{21'}$$

From (7) and (15) we obtain

$$C'_{\alpha\beta} = \sum_r B_{r\alpha} W_r B_{r\beta}. \tag{22'}$$

Inserting (22′) into (20′) we obtain

$$i[L_{ik}, W_r] = 0. \tag{23'}$$

From (6′) we have

$$C_{st} = \sum_r A_{sr} W_r A_{tr}. \tag{24'}$$

Because of (23′),

$$i[L_{ik}, C_{ss}] = 2\sum_r A_{ir} W_r A_{sr}\delta_{sk} - 2\sum_r A_{kr} W_r A_{sr}\delta_{is}. \tag{25'}$$

Inserting (24) into (21) and comparing with (25′), we obtain finally

$$i[L_{ik}, A_{sr}] = A_{ir}\delta_{ks} - A_{kr}\delta_{is}. \tag{26'}$$

(ii) The angular momentum operator in isobaric spin space is given by

$$T_{\alpha\beta} = \sum_r (\varphi_{\alpha r}\pi_{\beta r} - \varphi_{\beta r}\pi_{\alpha r}). \tag{27'}$$

By a similar calculation we can prove

$$i[T_{\alpha\beta}, B_{mr}] = \delta_{\beta m} B_{\alpha r} - \delta_{\alpha m} B_{\beta r}.$$

(iii) Let

$$L^{rs} = \sum_{i,k} A_{ir} A_{ks} L_{ik}, \qquad T^{rs} = \sum_{\alpha,\beta} B_{r\alpha} B_{s\beta} T_{\alpha\beta}. \tag{28'}$$

Using (26′) we obtain

$$i[L^{rs}, A_{lt}] = \sum_{i,k} A_{ir} A_{ks} [L_{ik}, A_{lt}] = \delta_{rt} A_{ls} - \delta_{st} A_{lr}. \tag{29'}$$

Similarly from (28′) we obtain

$$i[T^{rs}, B_{kt}] = \sum_{\alpha,\beta} B_{r\alpha} B_{s\beta} [T_{\alpha\beta}, B_{kt}] = \delta_{rt} B_{sr} - \delta_{st} B_{r\alpha}. \tag{30'}$$

From (19′), (27′), (29′) and (30′) we obtain

$$L^{rs} = \sum_{k,\alpha} (Q_r A_{ks} B_{r\alpha}\pi_{\alpha k} - Q_s A_{kr} B_{s\alpha}\pi_{\alpha k}), \tag{31'}$$

$$T^{rs} = \sum_{k,\alpha} (Q_r A_{kr} B_{s\alpha}\pi_{\alpha k} - Q_s A_{ks} B_{r\alpha}\pi_{\alpha k}) \tag{32'}$$

which give, when $r \neq s$,

$$\sum_{\alpha,k} A_{kr} B_{s\alpha}\pi_{\alpha k} = \tfrac{1}{2}\left\{\frac{L^{rs}+T^{rs}}{Q_r - Q_s} - \frac{L^{rs}-T^{rs}}{Q_r+Q_s}\right\}; \tag{33'}$$

and when $r = s$, we may define

$$\sum_{\alpha, k} A_{kr} B_{r\alpha} \pi_{\alpha k} = P_r. \tag{34'}$$

From (33') and (34') we obtain finally

$$\pi^{(0)*}_{\alpha s} \equiv \pi_{\alpha s} = \sum_{r, k} A_{sr} B_{k\alpha} \left\{ P_r \delta_{rk} + \tfrac{1}{2} \left(\frac{L^{rk} + T^{rk}}{Q_r - Q_k} - \frac{L^{rk} - T^{rk}}{Q_r + Q_k} \right) (1 - \delta_{rk}) \right\}. \tag{35'}$$

Appendix 2

In the following we shall investigate the effect of neglecting terms containing R_{kr} in eq. (26). We consider the term proportional to $(\sum_r \tau'_r \sigma'_r Q_r)^n$ of (37):

$$T_n = A \frac{1}{n!} (-g)^n (\tfrac{1}{2} I)^{\frac{1}{2}n} \left(\sum_{r=1}^{3} \tau'_r \sigma'_r Q_r \right)^n. \tag{36'}$$

We first evaluate the result of applying the terms of eq. (26) containing R_{12} to the expression (36'). Expanding (36') in the form

$$T_n = A (-g)^n (\tfrac{1}{2} I)^{\frac{1}{2}n} \sum_{n_3=0}^{n} \frac{1}{(n-n_3)!\, n_3!} (\tau'_1 \sigma'_1 Q_1 + \tau'_2 \sigma'_2 Q_2)^{n-n_3} (\tau'_3 \sigma'_3 Q_3)^{n_3} \tag{37'}$$

and using (28) we obtain

$$\begin{aligned} &\frac{g I^{\frac{1}{2}}}{2\sqrt{2}} (\tau'_2 \sigma'_1) R_{12} T_n \\ &= -A (-g)^{n+1} (\tfrac{1}{2} I)^{\frac{1}{2}(n+1)} (\tau'_2 \sigma'_1) \times \sum_{n_3=0}^{n-1} \sum_{l=0}^{n-n_3-1} \frac{1}{(n-n_3)!\, n_3!} \\ &\quad \cdot (\tau'_1 \sigma'_1 Q_1 + \tau'_2 \sigma'_2 Q_2)^{n-n_3-l+1} (\tau'_2 \sigma'_1) (\tau'_1 \sigma'_1 Q_1 + \tau'_2 \sigma'_2 Q_2)^l (\tau_3 \sigma_3 Q_3)^{n_3}. \end{aligned} \tag{38'}$$

Since $\tau'_2 \sigma'_1$ anti-commutes with $(\tau'_1 \sigma'_1 Q_1 + \tau'_2 \sigma'_2 Q_2)$ we may displace the factor $\tau'_2 \sigma'_1$ by using this commutation relation to the right. After adding the result obtained by interchanging 1 and 2, we obtain

$$\begin{aligned} &\frac{g}{2\sqrt{2}} I^{\frac{1}{2}} (\tau'_2 \sigma'_1 R_{12} + \tau'_1 \sigma'_2 R_{21}) T_n = -A (-g)^{n+1} (\tfrac{1}{2} I)^{\frac{1}{2}(n+1)} \\ &\times \sum_{n_3=0}^{n-1} [1 - (-1)^{n-n_3}] \frac{1}{(n-n_3)!\, n_3!} (\tau'_1 \sigma'_1 Q_1 + \tau'_2 \sigma'_2 Q_2)^{n-n_3-1} (\tau'_3 \sigma'_3 Q_3)^{n_3}. \end{aligned} \tag{39'}$$

On the other hand, inserting (37') into (30), we see that the first two terms of (30) operating on (37') give zero. The part containing $\tau'_3 \sigma'_3$ in the third term of (30) when applied to (36') gives

$$\frac{g}{\sqrt{2}} I^{\frac{1}{2}} \tau'_3 \sigma'_3 \frac{\partial}{\partial Q} T_n = A \frac{1}{(n-1)!} (-g)^{n+1} (\tfrac{1}{2}I)^{\frac{1}{2}(n+1)} \left(\sum_{r=1}^{3} \tau'_r \sigma'_r Q_r \right)^{n-1}$$

$$= A(-g)^{n+1} (\tfrac{1}{2}I)^{\frac{1}{2}(n+1)} \sum_{n_3=0}^{n-1} \frac{1}{(n-n_3-1)!\, n_3!} \tag{40'}$$

$$\cdot (\tau'_1 \sigma'_1 Q_1 + \tau'_2 \sigma'_2 Q_2)^{n-n_3-1} (\tau'_3 \sigma'_3 Q_3)^{n_3}.$$

We see that the coefficient of $(\tau'_1\sigma'_1Q_1+\tau'_2\sigma'_2Q_2)^{n-n_3-1}(\tau'_3\sigma'_3Q_3)^{n_3}$ in (39′) is smaller than the corresponding coefficient in (40′) by a factor $[1-(-1)^{n-n_3}]/(n-n_3)$. When $n-n_3$ is even this factor is zero; when $n-n_3$ is odd this factor is $2/(n-n_3)$. When n is very large, the terms in (39′) and (40′) which represent the largest probability are given by $n_3 \approx n-n_3 \approx \frac{1}{2}n$. Therefore the ratio of terms in (39′) and (40′) corresponding to the largest probability is equal to zero or $4/n$ This suffices to show that when n is very large, (39′) may be neglected in comparison to (40′). In other words the terms containing R_{rk} in (26) may be neglected when n is very large.

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UNTERSUCHUNGEN ÜBER QUANTENTHEORIEN MIT INDEFINITER METRIK

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Abstract: The conditions are examined under which a quantum theory with indefinite metric is equivalent to one with definite metric. The answer to this question depends on general properties of the physical states belonging to that theory. We call a physical state *proper*, if there exists at least one set of measurements the result of which individuates it completely; otherwise the physical state is called *improper*. It is necessary and sufficient for the existence of a physically equivalent quantum theory with definite metric, that there be proper physical states only. This result contains the theorem of Ascoli and Minardi [1]. There is no form with definite metric of a quantum theory having some improper states among its physical states. In such a theory, one finds, besides an effect already described in ref. [7]) that transition probability is not always symmetric.

1. Einleitung

Das mögliche Auftreten indefiniter Metrik in Quantentheorien wurde bereits 1942 von Dirac [2]) in Erwägung gezogen und wurde seitdem oft untersucht. Auf Grund der Möglichkeit, für das Modell von Lee [6]) exakte und geschlossene Lösungen zu finden, war dessen Analyse durch Källen und Pauli [5]) sowie durch Heisenberg [4]) besonders wichtig. Sie zeigte u.a., daß die von Lee gegebenen Operatorgleichungen mit Notwendigkeit auf eine indefinite Metrik führen.

Ascoli und Minardi [1]) waren wohl die ersten, die eine neue Seite des Problems zu untersuchen begannen, indem sie die allgemeine Frage stellten: Gegeben sei eine konsistente Quantentheorie mit nicht definiter Metrik. Kann man diese Theorie auf den Fall des Vorliegens definiter Metrik zurückführen? Ascoli und Minardi bejahen diese Frage. Andererseits wurde in Ref. [7]) ein noch sehr skizzenhafter Versuch des Aufbaus einer in diesem Sinne nicht "zurückführbaren" Quantentheorie unternommen. Die nähere Untersuchung, die in dieser Arbeit vorgenommen wird, zeigt, daß sich diese beiden Ergebnisse nicht widersprechen.

Tatsächlich sind die von Ascoli und Minardi gewählten Voraussetzungen scharf genug, um mit Hilfe eines genügend weit gefaßten Äquivalenzbegriffes die Zurückführbarkeit einer diesen Voraussetzungen genügenden Quantentheorie auf eine Theorie mit positiv definiter Metrik zu gewährleisten. Wir zeigen, daß erstens unter sehr allgemeinen Voraussetzungen die

Elimination der unphysikalischen Zustände stets möglich ist, und daß zweitens dieser Prozeß zu einer Quantentheorie mit definiter Metrik führt — vorausgesetzt, der Begriff des Zustandes eines Systems wird geeignet gefaßt.

Nennen wir einen physikalischen Zustand *eigentlich*, wenn mindestens ein System gleichzeitig meßbarer Observablen existiert, deren Messung diesen Zustand völlig charakterisiert, so können wir das erwähnte Resultat von Ascoli und Minardi wiefolgt charakterisieren: Beschreibt eine Quantentheorie nur eigentliche physikalische Zustände, so ist sie einer Quantentheorie mit definiter Metrik äquivalent.

Umgekehrt kann man *die* Quantentheorien mit indefiniter Metrik, die keine Rückführung auf eine Theorie mit definiter Metrik gestatten, durch das Auftreten uneigentlicher physikalischer Zustände völlig charakterisieren. Bei solchen Theorien treten neben einer schon in Ref. [7]) beschriebenen Abweichung Fälle von nichtsymmetrischer Übergangswahrscheinlichkeit auf. D.h. in solchen Theorien gibt es stets physikalische Zustände mit

$$W_{A\to B} \neq W_{B\to A}.$$

Methodisch schließt sich unsere Arbeit an die Grundidee von Minardi und Ascoli an, daß die Quantentheorien mit indefiniter Metrik axiomatisch in ihrer Grundstruktur zu erfassen sind und somit einer einheitlichen Axiomatik unterliegen müssen. Ein bemerkenswerter Schluß von Ascoli und Minardi ermöglicht es, dieses Programm im Sinne der berühmten Diracschen Axiomatik [3]) in Angriff zu nehmen: Bei der Formulierung einer Feldquantentheorie hat man im allgemeinen nicht nur Operatoren, die Observablen zugeordnet sind, zu berücksichtigen. Solche nichthermitischen Operatoren (z.B. Erzeugungsoperatoren) spielen beim Aufbau des Hilbert-Raumes eine entscheidende Rolle. Um jedoch die Theorie mit dem Experiment zu vergleichen, benötigen wir nur Operatoren, die Observablen entsprechen und deshalb ist für die Verbindung Theorie—Experiment nur wichtig, daß die Wirkung *dieser* Operatoren auf die Zustandsvektoren erklärt ist. Eine Quantenfeldtheorie mag deshalb in Bezug auf die Gesamtheit aller betrachteten Operatoren irreduzibel (siehe Abschnitt 3) sein, aber reduzibel bei Beschränkung auf die den beobachtbaren Größen zugeordneten Operatoren. Für die allgemeine Untersuchung können wir also gleich annehmen, daß der Hilbert-Raum der Zustände eine Darstellung derjenigen Operatoren ist, die den Observablen entsprechen ("physikalische Operatoren"), ohne uns darum zu kümmern, ob die Wirkung anderer Operatoren auf diesen Hilbert-Raum erklärt bezw. erklärbar ist. Wir stellen uns deshalb auf den Standpunkt, daß das physikalisch Wichtige einer Quantentheorie die *physikalischen* Operatoren und ihre Darstellung in einem Hilbert-Raum, der die möglichen Zustände des Systems beschreibt, sind.

2. Das mathematische Gerüst einer Quantentheorie

Das mathematische Gerüst einer Quantentheorie besteht aus zwei Bestandteilen.

Der erste Bestandteil ist ein System von Operatoren $\alpha, \alpha_1, \ldots, \beta, \beta_1, \ldots$, die den beobachtbaren Größen zugeordnet sind und zwischen denen gewisse Beziehungen bestehen, die wir durch

$$\begin{aligned} f_1(\alpha, \alpha_1, \ldots, \beta, \ldots) &= 0, \\ f_2(\alpha, \alpha_1, \ldots, \beta, \ldots) &= 0, \text{ usw.} \end{aligned} \tag{1}$$

andeuten wollen. Hierdurch ist eine Operatoralgebra R über den komplexen Zahlen gegeben, die von den Operatoren $\alpha, \alpha_1, \ldots, \beta, \ldots$ erzeugt und durch die Relationen (1) bestimmt (definiert) ist. Die den beobachtbaren Größen zugeordneten Operatoren $\alpha, \ldots$ nennen wir *physikalische* Operatoren, zum Unterschied zu irgendwelchen anderen Elementen von R.

Da wir die physikalischen Operatoren später als hermitesche Operatoren deuten wollen, muß in R die durch

$$\text{a)} \quad (\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^*, \quad (c_1\gamma_1+c_2\gamma_2)^* = c_1^*\gamma_1^*+c_2^*\gamma_2^* \tag{2}$$

für beliebige $\gamma_1, \gamma_2 \in \mathrm{R}$

$$\text{b)} \quad \alpha = \alpha^* \text{ für jeden physikalischen Operator}$$

bestimmte Involution existieren (was den Relationen (1) gewisse Beschränkungen auferlegt). Hierbei sind die c_k komplexe Zahlen und die c_k^* die zu ihnen konjugierten.

Der zweite Bestandteil des mathematischen Gerüstes ist ein Hilbert-Raum Z, der die Algebra R involutionstreu darstellt und die möglichen physikalischen Zustände eines quantentheoretischen Systems beschreibt. Wir verstehen dabei den Begriff "Hilbert-Raum" in jenem erweiterten Sinn, der die Möglichkeit einer indefiniten Metrik einschließt. Das Wort „involutionstreu" bedeutet, daß mit der in R erklärten Involution (2)

$$\langle \mathrm{A}|\gamma^*|\mathrm{B}\rangle = \langle \mathrm{B}|\gamma|\mathrm{A}\rangle^*$$

für beliebige γ aus R und Vektoren $|\mathrm{A}\rangle$, $|\mathrm{B}\rangle$ aus Z gilt.

3. Reduzible und irreduzible Quantentheorien

Wir nennen eine Quantentheorie, die die Operatoralgebra R und den Darstellungsraum Z als mathematisches Gerüst besitzt *reduzibel*, wenn ein Teilraum Z_0 von Z folgender Beschaffenheit existiert:

a) Z_0 ist eine Darstellung von R, d.h. $\mathrm{R} \cdot \mathrm{Z}_0 = \mathrm{Z}_0$, die nicht nur aus der Null von Z besteht.

b) Ist Z_1 die Menge aller zu Z_0 orthogonalen Vektoren, so gibt es Vektoren $|\mathrm{A}\rangle$, $|\mathrm{B}\rangle$ in Z_1 mit

$$\langle \mathrm{A}|\mathrm{B}\rangle \neq 0.$$

Erfüllt nun Z_0 diese Bedingungen, so gilt auch für den Teilraum Z_1 die Formel $R \cdot Z_1 = Z_1$; denn ist $\langle A|B\rangle = 0$ für alle $|B\rangle \in Z_0$ und $\gamma \in R$ beliebig, so ist auch $\langle A|\gamma^*|B\rangle = 0$ wegen a) und deshalb ist

$$\langle B|\gamma|A\rangle = \langle A|\gamma^*|B\rangle^* = 0 \quad \text{für alle } |B\rangle \in Z_0,$$

d.h. $\gamma|A\rangle \in Z_1$.

Es folgt hieraus, daß mit Z_0 und Z_1 auch ihr mengentheoretischer Durchschnitt $Z_0 \cap Z_1$ eine involutionstreue Darstellung von R ist. Eine weitere involutionstreue Darstellung erhält man deshalb, wenn man in Z_1 modulo $Z_1 \cap Z_0$ rechnet. Es entsteht dann durch Restklassenbildung eine Darstellung †

$$Z_1/Z_1 \cap Z_0$$

die in gewissem Sinn „weniger" reduzibel ist als Z_1 oder gar Z.
Man wird also erwarten, daß durch den eben skizzierten Prozeß aus einer reduziblen eine irreduzible Theorie gewinnbar ist. Wir wollen im Folgenden stets nur irreduzible Theorien betrachten ††.

Die Metrik des Hilbert-Raumes Z einer irreduziblen Theorie ist nicht entartet (nicht singulär); denn die Menge Z_0 aller $|B\rangle$, für die $\langle A|B\rangle = 0$ für alle $|A\rangle$ aus Z ist, erfüllt die Bedingungen a) und b) dieses Abschnittes †††.

4. Die Zuordnung der Vektoren des Hilbert-Raumes zu den physikalischen Zuständen

Sei R die Operatoralgebra und Z der Dastellungsraum einer irreduziblen Quantentheorie. Wir wollen uns mit der Frage beschäftigen, auf welche Weise den Vektoren aus Z physikalische Zustände zugeordnet werden können.

Sei $S = (\alpha_1, \alpha_2, \ldots)$ ein System kommutierender physikalischer Operatoren. Wir nennen es genau dann *vollständig*, wenn folgender Sachverhalt vorliegt: Sind $\lambda_1, \lambda_2, \ldots$ reelle Zahlen, so gibt es höchstens einen physikalischen Zustand, der bei simultaner Messung der $\alpha_1, \alpha_2, \ldots$ genau die Werte $\lambda_1, \lambda_2, \ldots$ liefert. Diesen durch S und die Werte $\lambda_1, \lambda_2, \ldots$ eindeutig bestimmten physikalischen Zustand bezeichnen wir mit $S(\lambda_1, \lambda_2, \ldots)$, vorausgesetzt, daß ein solcher überhaupt existiert *. Für den Sachverhalt „S ist ein vollständiges System kommutierender physikalischer Operatoren" sagen wir kürzer „S ist ein *vollständiger Satz.*"

† Für zwei Vektoren aus Z_1 setzen wir also genau dann $|A\rangle = |B\rangle$, wenn $|A\rangle - |B\rangle$ in $Z_1 \cap Z_0$ liegt. Die Metrik verschwindet auf $Z_1 \cap Z_0$ identisch; denn $Z_1 \cap Z_0$ ist zu sich selbst orthogonal. Kongruente Vektoren aus Z_1 besitzen deshalb gleiche Norm.

†† Es gibt somit keinen echten Teilraum von Z, der bereits eine Darstellung von R wäre.

††† Die Metrik $\langle A|B\rangle$ heißt *entartet*, wenn ein $|A\rangle$ mit $|A\rangle \neq 0$ existiert mit $\langle A|B\rangle = 0$ für alle $|B\rangle$.

* Hierdurch soll nicht gesagt werden, daß sich *jeder* physikalische Zustand auf diese Weise charakterisieren läßt (siehe Abschnitt 7).

Axiom I: Ist S ein vollständiger Satz, so existiert zu jedem physikalischen Zustand $S(\lambda_1, \lambda_2, \ldots)$ ein Vektor $|A\rangle$ in Z mit

$$\alpha_K|A\rangle = \lambda_K|A\rangle, \qquad \alpha_K \in S.$$

Ist $|B\rangle$ ein zweiter Vektor dieser Beschaffenheit, so ist

$$|B\rangle = c \cdot |A\rangle$$

mit einer komplexen Zahl c.

Die gemäß Axiom I zu den physikalischen Zuständen $S(\lambda_1, \lambda_2, \ldots)$ gehörenden Vektoren aus Z mögen die *physikalischen* S-*Eigenvektoren* genannt werden.

Satz 1: Sei S ein vollständiger Satz und $|A\rangle$ ein physikalischer S-Eigenvektor. Dann ist $\langle A|A\rangle \neq 0$. Ist $|B\rangle$ ein anderer (nicht notwendig physikalischer) Eigenvektor von S, so ist entweder $\langle A|B\rangle = 0$ oder es ist $|B\rangle = c \cdot |A\rangle$ mit einer komplexen Zahl c.

Zum Beweis nehmen wir $|B\rangle \neq c \cdot |A\rangle$ an. Es gibt dann einen zu S gehörenden Operator α mit

$$\alpha|A\rangle = \lambda \cdot |A\rangle, \quad \alpha|B\rangle = \mu \cdot |B\rangle, \quad \lambda \neq \mu.$$

Aus $\langle A|\alpha|B\rangle = \langle B|\alpha|A\rangle^*$ folgt dann $(\mu - \lambda^*)\ \langle A|B\rangle = 0$. Da $|A\rangle$ nach Voraussetzung ein physikalischer S-Eigenvektor sein sollte, ist λ reell und somit folgt $\langle A|B\rangle = 0$.

Jeder Vektor $|C\rangle$ besitzt nun eine Entwicklung

$$|C\rangle = c \cdot |A\rangle + \sum |B_k\rangle,$$

wobei die $|B_k\rangle$ von $|A\rangle$ unabhängige S-Eigenvektoren (nicht notwendig physikalische) sind. Es folgt $\langle A|C\rangle = c \cdot \langle A|A\rangle$. Wäre $\langle A|A\rangle = 0$, so wäre die Metrik auf Z entartet. Dies aber widerspricht der Voraussetzung, daß es sich um eine irreduzible Quantentheorie handelt.

5. Das Superpositionsprinzip

Ist S ein vollständiger Satz, so bezeichnen wir mit Z(S) den von den physikalischen S-Eigenvektoren aufgespannten Teilraum von Z und mit $Z_0(S)$ den Raum der zu Z(S) orthogonalen Vektoren.

Aus Satz 1 folgt, daß $Z_0(S)$ von den nicht-physikalischen S-Eigenvektoren aufgespannt wird. Deshalb ist

$$Z = Z(S) + Z_0(S), \quad Z(S) \cap Z_0(S) = 0. \tag{3}$$

Hieraus folgt, da Z(S) und $Z_0(S)$ zueinander orthogonal sind, daß die Beschränkung der Metrik auf Z(S) bzw. auf $Z_0(S)$ zu einer nicht ausgearteten Metrik in Z(S) bezw. $Z_0(S)$ führt. Natürlich haben wir Z(S) in Hinblick auf das Superpositionsprinzip gebildet: Da Z(S) von physikalischen S-Eigenvektoren erzeugt wird, wird jeder Vektor aus Z(S) einen gewissen physikalischen Zustand beschreiben.

Axiom II: Ist S ein vollständiger Satz, so entspricht jedem von Null verschiedenen Vektor aus Z(S) genau ein physikalischer Zustand und umgekehrt jedem physikalischen Zustand bis auf einen konstanten Faktor genau ein Vektor aus Z(S).

Betrachten wir zwei vollständige Sätze S_1 und S_2. Im allgemeinen Fall wird dann $Z(S_1) \neq Z(S_2)$ sein. Nach Axiom II gehört zu jedem physikalischen Zustand sowohl ein Vektor $|A_1\rangle$ aus $Z(S_1)$ als auch ein Vektor $|A_2\rangle$ aus $Z(S_2)$. Es muß folglich eine Abbildung τ_{21} von $Z(S_1)$ auf $Z(S_2)$ existieren, die $|A_1\rangle$ in $|A_2\rangle$ überführt:

$$|A_1\rangle \to |A_2\rangle = \tau_{21}|A_1\rangle. \tag{4}$$

Damit das Superpositionsprinzip gewahrt bleibt, muß für diese Abbildung

$$\tau_{21}(c \cdot |A_1\rangle + d \cdot |B_1\rangle) = c \cdot |A_2\rangle + d \cdot |B_2\rangle \tag{4a}$$

gelten. Eine solche Zuordnung gibt es in der Tat:

Ist $|A_1\rangle$ aus $Z(S_1)$, so gibt es wegen Formel (3) genau eine Zerlegung

$$|A_1\rangle = |A_2\rangle + |A_2^0\rangle, \quad |A_2\rangle \in Z(S_2), \quad |A_2^0\rangle \in Z_0(S_2) \tag{5}$$

und die Definition

$$\tau_{21}|A_1\rangle = |A_2\rangle \tag{5a}$$

erfüllt die Bedingung (4a).

Nun muß die obige Zuordnung offenbar *symmetrisch* sein: Die analog definierte Abbildung τ_{12} von $Z(S_2)$ auf $Z(S_1)$ muß bis auf einen konstanten Faktor zu τ_{21} reziprok sein

$$\tau_{12} \cdot \tau_{21} = \text{const.} \tag{6}$$

Dies ist genau dann der Fall, wenn aus (5)

$$|A_2\rangle = c \cdot |A_1\rangle + |A_1^0\rangle, \quad |A_1^0\rangle \in Z_0(S_2), \quad c = \text{feste Zahl} \tag{5b}$$

folgt; denn dann ist

$$\tau_{12} \cdot \tau_{21} = c. \tag{6a}$$

Noch schärfer muß man fordern, daß die Abbildungen zwischen den Z(S) *transitiv* sind: Liegen drei vollständige Sätze S_i vor und sind τ_{rs} die nach (5a) definierten Abbildungen, so muß

$$\tau_{12} \cdot \tau_{23} \cdot \tau_{31} = \text{const.} \tag{7}$$

gelten.

Die Bedingungen (6) und (7) sind nicht nur notwendig, sondern auch hinreichend für die Gewährleistung des Superpositionsprinzips gemäß Axiom II. Mehr noch, aus (6) bzw. (6a) folgt automatisch

$$\langle \tau_{21} A_1 | \tau_{21} B_1 \rangle = c \cdot \langle A_1 | B_1 \rangle; \qquad |A_1\rangle, |B_1\rangle \in Z(S_1). \tag{8}$$

Aus (5) bzw. (5b) folgt nämlich

$$\langle A_2 | A_1 \rangle = \langle A_2 | A_2 \rangle \quad \text{bzw.} \quad \langle A_1 | A_2 \rangle = c \cdot \langle A_1 | A_1 \rangle.$$

Hieraus folgt Formel (8) zunächst für den Fall $|A_1\rangle = |B_1\rangle$, aus dem der allgemeine durch Ausrechnung von $|A_1\rangle + d \cdot |B_1\rangle$ in bekannter Weise folgt. Bemerken wir noch, daß $\langle A_2|A_1\rangle$ reell ist und deshalb auch $\langle A_1|A_2\rangle$ und schließlich c reell sein wird.

Betrachten wir kurz den Fall, daß die Metrik auf einem Raum $Z(S)$ *definit* ist. Dann ist wegen (8) die Metrik auf jedem Raum $Z(S_1)$ auch definit. Seien nun zwei physikalische Zustände gegeben, die sich in $Z(S)$ durch $|A\rangle$ und $|B\rangle$, in $Z(S_1)$ aber durch $|A_1\rangle$ und $|B_1\rangle$ darstellen lassen. Es gibt dann eine von der Wahl des vollständigen Satzes S unabhängige Übergangswahrscheinlichkeit; denn aus (8) folgt

$$\left|\frac{\langle A|B\rangle^2}{\langle A|A\rangle \cdot \langle B|B\rangle}\right| = \left|\frac{\langle A_1|B_1\rangle^2}{\langle A_1|A_1\rangle \cdot \langle B_1|B_1\rangle}\right|. \tag{9}$$

Obwohl also die Theorie irreduzibel ist und unphysikalische Zustände existieren mögen, kann bei Gültigkeit der Bedingungen (6) und (7) eine allen Anforderungen genügende Übergangswahrscheinlichkeit definiert werden — vorausgesetzt, die Beschränkung der Metrik auf die $Z(S)$ ist definit.

Die Tatsache, daß die Teilräume $\bar{Z}(S)$ wegen (8) im wesentlichen dieselbe metrische Struktur besitzen, legt die Frage nahe, ob sich die unphysikalischen Vektoren $Z_0(S)$ auf irgendeinem Weg aus der Theorie entfernen lassen. Aus unserer Voraussetzung nach Irreduzibilität der Theorie folgt, daß man den Gesamtraum Z nur dann verändern kann, wenn man die zwischen den physikalischen Operatoren $\alpha_1\ \alpha_1, \ldots$, bestehenden Beziehungen (1) gleichzeitig in *nicht-trivialer* Weise verändert: während es sich bei der im Abschnitt 3 erwähnte Reduktion um einen Übergang †

$$R, Z \to R, Z'$$

handelt, muß die Ausschaltung der unphysikalischen Zustände mit Notwendigkeit ein Übergang

$$R, Z \to R', Z'$$

sein. Wir nennen die beiden Quantentheorien R, Z und R', Z' *physikalisch äquivalent*, wenn aus ihnen dieselben physikalischen Folgerungen gezogen werden können.

6. Das „Ausfegen" der nicht-physikalischen Zustände

Wir beweisen nun den Satz:

Satz 2: Zu jeder Quantentheorie R, Z, die den Axiomen I und II genügt, existiert eine physikalisch äquivalente Theorie R', Z', die ebenfalls diesen Axiomen genügt und die keine unphysikalischen Vektoren besitzt. D.h. ist

† Wir betonen, daß durch die Reduktion (Abschnitt 3) die Relationen zwischen die *physikalischen* Operatoren bestehen bleiben. Über Operatoren, denen *keine* Observablen entsprechen, ist damit nichts gesagt.

S′ irgend ein vollständiger Satz, so ist

$$Z' = Z'(S').$$

Unter etwas schärferen Voraussetzungen wurde ein ähnlicher Satz von Ascoli und Minardi bewiesen, die allerdings nicht die Möglichkeit $Z(S) \neq Z(S_1)$ in Betracht zogen und deshalb mit dem in Abschnitt 3 erwähnten Reduktionsprozeß auskommen. Siehe auch den nächsten Abschnitt.

Zum Beweis wählen wir uns einen vollständigen Satz S_0 aus und setzen $Z' = Z(S_0)$.

Ist nun S_1 ein beliebiger anderer vollständiger Satz und ist der physikalische Operator α in S_1 enthalten, so definieren wir mit Hilfe der im Abschnitt 5 erklärten Abbildungen τ den Operator

$$\alpha' = \tau_{10}^{-1} \alpha \tau_{10}, \tag{11}$$

der offensichtlich $Z(S_0)$ d.h. Z' in sich überführt. Wegen (8) ist α' hermitesch in Z'. Jedem physikalischen Operator α darf natürlich nur ein Operator α'. entsprechen. Sind also S_1 und S_2 vollständige Sätze und ist der Operator α sowohl in S_1 als auch in S_2, so muß

$$\tau_{10}^{-1} \alpha \tau_{10} = \tau_{20}^{-1} \alpha \tau_{20}$$

gelten. D.h. der Operator α muß mit dem Operator $\tau_{10} \tau_{20}^{-1}$ kommutieren. Wegen Formel (6) und (7) muß also α mit τ_{21} kommutieren. Dies zeigen wir folgendermaßen. Ist $|A_1\rangle$ in $Z(S_1)$, so ist wegen (5) $\tau_{21}|A_1\rangle$ durch die Forderung

$$|A_1\rangle - \tau_{21}|A_1\rangle \in Z_0(S_2), \quad \tau_{21}|A_1\rangle \in Z(S_2)$$

eindeutig bestimmt. Da α zu S_2 gehört, läßt es sowohl $Z(S_2)$ als auch $Z_0(S_2)$ invariant, und somit ist

$$\alpha|A_1\rangle - \alpha\tau_{21}|A_1\rangle \in Z_0(S_2), \quad \alpha\tau_{21}|A_1\rangle \in Z(S_2).$$

Andererseits ist definitionsgemäß

$$\alpha|A_1\rangle - \tau_{21}\alpha|A_1\rangle \in Z_0(S_2), \quad \tau_{21}\alpha|A_1\rangle \in Z(S_2)$$

und der Vergleich liefert

$$\tau_{21}\alpha|A_1\rangle = \alpha\tau_{21}|A_1\rangle \quad \text{für alle } |A_1\rangle \in Z(S_1).$$

Somit hängt die Zuordnung (11) nur vom physikalischen Operator α ab und nicht davon, welchen vollständigen, α enthaltenden Satz man zur Bestimmung von α' wählt.

Indem man nun alle physikalischen Operatoren α gemäß der Vorschrift (11) abändert, erhält man eine Theorie, die Z' als Darstellungsraum besitzt und *keine* unphysikalischen Vektoren besitzt. Diese Überführung läßt alle physikalischen Folgerungen unangetastet:

a) Wegen (11) sind die zu physikalischen Zuständen gehörenden Eigenwerte von α wiederum Eigenwerte von α'.

b) Jedem vollständigen Satz S entspricht gemäß (11) ein vollständiger Satz S'.

c) Wegen (8) werden die metrischen Beziehungen zwischen den physikalischen Vektoren aus Z(S), S beliebig, bis auf triviale Abänderungen in Z' reproduziert. Dies trifft wegen (9) insbesondere auf die Ausdrücke

$$\left| \frac{\langle A|B\rangle^2}{\langle A|A\rangle\langle B|B\rangle} \right|$$

zu.

Was jedoch *nicht* erhalten bleibt ist offensichtlich die Gesamtheit der Relationen (1), da der in (11) vorkommende Operator τ_{10} von α abhängt! Es entstehen deshalb *neue* Relationen

$$\begin{aligned} f'_1(\alpha', \alpha'_1, \ldots, \beta', \ldots) &= 0, \\ f'_2(\alpha', \alpha'_1, \ldots, \beta', \ldots) &= 0, \end{aligned} \tag{12}$$

usw.

Jedoch kann man auch hier noch eine allgemeine Aussage machen:

d) Besteht zwischen den physikalischen Operatoren $\alpha_1, \alpha_2, \ldots$ eine Beziehung $f(\alpha_1, \alpha_2, \ldots) = 0$ und kommutieren die Operatoren $\alpha_1, \alpha_2, \ldots$ untereinander, so gilt dieselbe Beziehung auch zwischen den neuen Operatoren $\alpha'_1, \alpha'_2, \ldots$

Eine letzte Bemerkung betrifft die Abhängigkeit der Theorie R', Z' von der Wahl $Z' = Z(S_0)$. Wegen (6) und (7) kann nachgewiesen werden, daß bei einer anderen Wahl $Z'' = Z(S_1)$ und analoger Definition von R'' die Abbildung τ_{10} die Eigenschaft $R'' = \tau_{10} R' \tau_{10}^{-1}$ besitzt und sich folglich die beiden Theorien nur in trivialer Weise unterscheiden.

7. „Eigentliche" und „uneigentliche" physikalische Zustände

Wegen Satz 2 können wir nun ohne Beschränkung der Allgemeinheit annehmen, daß R, Z eine irreduzible Theorie *ohne nicht-physikalische Zustände* ist. Insbesondere ist dann

$$Z = Z(S) \text{ für jeden vollständigen Satz S.} \tag{13}$$

Der Normalfall wird dann sein, daß Z *definite* Metrik trägt. Wann kann man dies mit Sicherheit garantieren und wann nicht?

Ist $|A\rangle \in Z$ ein Vektor, so entspricht ihm nach Axiom II und wegen obigen Voraussetzungen ein physikalischer Zustand. Wir nennen diesen Zustand *eigentlich* (proper), wenn mindestens ein vollständiger Satz S existiert, so daß $|A\rangle$ ein S-Eigenvektor ist. Im anderen Fall nennen wir $|A\rangle$ *uneigentlich* (improper). Wir können diese Definition auch so wenden:

Liegt ein uneigentlicher Zustand vor, so enthält *jeder* vollständige Satz mindestens eine Observable, die beim Vorliegen dieses Zustandes keinen

scharfen Wert annimmt. D.h. der Unterschied zwischen eigentlichen und uneigentlichen Zuständen ist im Prinzip beobachtbar.

Satz 3: Zu jeder Quantentheorie, die den Axiomen I und II genügt und die nur eigentliche physikalische Zustände beschreibt, gibt es eine physikalisch äquivalente Theorie mit definiter Metrik.
Dies ist das wesentliche Resultat von Ascoli und Minardi, die die physikalischen Zustände per Definition als eigentlich voraussetzen.

Der Beweis von Satz 3 ist leicht. Nach Satz 2 langt es, sich auf eine Theorie ohne nicht-physikalische Zustände zu beschränken. Besitzt eine solche Theorie nur eigentliche Zustände, so ist nach Satz 1 für jeden ihrer Vektoren $\langle A|A\rangle \neq 0$, $|A\rangle \neq 0$ vorausgesetzt. Dies ist aber nur bei definiter Metrik möglich. Es ist wert, auch die Umkehrung dieses Satzes zu vermerken:

Satz 4: Beschreibt eine Quantentheorie auch uneigentliche physikalische Zustände, so besitzt keine zu ihr physikalisch äquivalente Theorie definite Metrik.
Denn sonst hätten wir (eventuell nach Reduzierung und „Ausfegen" der nicht-physikalischen Zustände) eine Quantentheorie mit definiter Metrik und eine solche kennt nur eigentliche Zustände.

Betrachten wir nun Theorien mit uneigentlichen Zuständen. Wegen (13) folgt aus Abschnitt 5 bereits, daß sich *jeder* physikalische Zustand nach eigentlichen Zuständen "entwickeln" läßt. Genauer: Ist S ein vollständiger Satz und $|A_k\rangle$, $k = 1, 2, \ldots$ ein vollständiges System von S-Eigenvektoren, so besitzt jeder Vektor $|A\rangle$ eine Zerlegung

$$|A\rangle = \sum c_k |A_k\rangle, \tag{14a}$$

wobei wegen Satz 1 die Normierung

$$\langle A_k|A_k\rangle = \pm 1 \tag{14b}$$

vorausgesetzt werden darf. Wir werden dann die Wahrscheinlichkeit, bei der Messung des Zustandes $|A\rangle$ den Zustand $|A_k\rangle$ zu finden, proportional $|c_k|^2$ annehmen dürfen. Infolge der Normierungsvorschrift für Wahrscheinlichkeiten haben wir also

$$W_{|A\rangle \to |A_k\rangle} = \frac{|c_k|^2}{\sum |c_i|^2}. \tag{15a}$$

Um hierfür einen geschlosseneren Ausdruck anzugeben, definieren wir den vom vollständigen Satz S abhängigen Operator $\varepsilon = \varepsilon(S)$ durch,

$$\varepsilon|A_k\rangle = \begin{matrix} +|A_k\rangle & \text{falls} & \langle A_k|A_k\rangle = +1, \\ -|A_k\rangle & \text{falls} & \langle A_k|A_k\rangle = -1. \end{matrix}$$

Für diesen Operator gilt

$$\varepsilon^2 = 1, \quad \varepsilon^* = \varepsilon, \quad \varepsilon\alpha = \alpha\varepsilon \quad \text{für} \quad \alpha \in S, \tag{16}$$

und man weist leicht nach, daß ε durch S eindeutig bestimmt ist. Man sieht aus der Definition, daß die Operatoren $\frac{1}{2}(1\pm\varepsilon)$ Projektoren sind, die den Gesamtraum Z in einen Teilraum mit positiv und einen Teilraum mit negativ definiter Metrik zerlegen. Hiernach ist

$$\langle A|B\rangle_S = \langle A|\tfrac{1}{2}(1+\varepsilon)|B\rangle - \langle A|\tfrac{1}{2}(1-\varepsilon)|B\rangle = \langle A|\varepsilon|B\rangle \tag{17}$$

eine positiv definite Metrik in ganz Z. Da aus (15a)

$$c_k = \langle A_k|A\rangle_S, \quad \langle A_k|A_k\rangle_S = 1,$$
$$\langle A|A\rangle_S = \sum |c_k|^2$$

folgt, ist

$$W_{|A\rangle\to|A_k\rangle} = \frac{|\langle A|A_k\rangle_S|^2}{\langle A|A\rangle_S} = \frac{|\langle A|A_k\rangle_S|^2}{\langle A|A\rangle_S\langle A_k|A_k\rangle_S}. \tag{15b}$$

Damit haben wir folgende

Definition: Ist $|A\rangle$ ein Zustand und $|B\rangle$ ein eigentlicher Zustand (beide physikalisch), so ist die Übergangswahrscheinlichkeit $|A\rangle \to |B\rangle$ wiefolgt bestimmt: Ist S ein vollständiger Satz und $|B\rangle$ ein S-Eigenvektor, so ist

$$W_{|A\rangle\to|B\rangle} = \frac{|\langle A|B\rangle_S|^2}{\langle A|A\rangle_S\langle B|B\rangle_S}.$$

Enthält die Theorie uneigentliche Zustände, so gibt es mit Sicherheit zwei vollständige Sätze S_1 und S_2 mit $\varepsilon(S_1) \neq \varepsilon(S_2)$; denn sonst wäre entweder die Metrik definit oder die Theorie reduzibel, was den gemachten Voraussetzungen widerspricht (alle physikalischen Operatoren wären mit $\frac{1}{2}(\varepsilon(S_1)+1)$ vertauschbar). Hieraus folgt, daß keine Gleichung der Form

$$\langle A|B\rangle_{S_1} = \lambda \cdot \langle A|B\rangle_{S_2}, \qquad \lambda \text{ fest} \tag{+}$$

existiert. Wir können deshalb S_k-Eigenvektoren $|A_k\rangle$ so finden, daß

$$W_{A_2\to A_1} = \frac{|\langle A_2|A_1\rangle_{S_1}|^2}{\langle A_1|A_1\rangle_{S_1}\langle A_2|A_2\rangle_{S_1}} \neq \frac{|\langle A_1|A_2\rangle_{S_2}|^2}{\langle A_1|A_1\rangle_{S_2}\langle A_2|A_2\rangle_{S_2}} = W_{A_1\to A_2} \tag{++}$$

ist. Denn da die S_k-Eigenvektoren ein vollständiges Orthogonalsystem bilden, könnte man bei Nichtbestehen von (++) auf (+) schließen. Damit haben wir

Satz 5: In einer Theorie mit uneigentlichen physikalischen Zuständen gibt es stets eigentliche physikalische Zustände $|A\rangle$, $|B\rangle$ mit

$$W_{|A\rangle\to|B\rangle} \neq W_{|B\rangle\to|A\rangle}.$$

Betrachten wir nun nochmals die Entwicklung (14) und einen Operator α aus S. Der "Erwartungswert" $\bar{\alpha}$ wird dann

$$\bar{\alpha} = \frac{\sum \lambda_k |c_k|^2}{\sum |c_k|^2}, \quad \alpha|A_k\rangle = \lambda_k|A_k\rangle \tag{18a}$$

sein. Hieraus folgt leicht

$$\bar{\alpha}(S) = \frac{\langle A|\alpha|A\rangle_S}{\langle A|A\rangle_S}, \tag{18b}$$

wobei die Schreibweise $\bar{\alpha}(S)$ andeutet, daß nicht nur von α, sondern auch von S abhängen kann. Es ist also *möglich*, daß

$$\bar{\alpha}(S_1) \neq \bar{\alpha}(S_2) \quad \text{für} \quad \alpha \in S_1, \alpha \in S_2 \tag{19}$$

ist. D.h. es sind zwar nicht die wirklich gemessenen Werte (die Eigenwerte von α) bei der Messung von S_1 und S_2 verschieden, wohl aber ihre statistische Verteilung.

Satz 6: Liegt eine Quantentheorie mit uneigentlichen physikalischen Zuständen vor und gehört der physikalische Operator α zu zwei vollständigen Sätzen S_1 und S_2, so ist folgendes Verhalten möglich: Die Analyse eines Zustandes nach den Observablen S_1 liefert einen anderen Erwartungswert $\bar{\alpha}(S_1)$ als die Analyse nach den Observablen des vollständigen Satzes S_2. Während Satz 5 ein notwendiges und hinreichendes Kriterium für das Vorhandensein uneigentlichen Zustände liefert, ist die in Satz 6 ausgesprochene Erscheinung hierfür nur hinreichend aber nicht notwendig.

Herrn Prof. Dr. G. Heber möchte ich für anregende Diskussionen und für sein freundliches Interesse herzlich danken.

Anhang

DER HEISENBERGSCHE STREUOPERATOR

In einer irreduziblen Quantentheorie mit Operatoralgebra R und Darstellungsraum Z betrachten wir die Gleichung

$$i\hbar \frac{d}{dt}|A(t)\rangle = H(t)|A(t)\rangle \tag{20}$$

mit physikalischen Operatoren $H(t)$ aus R, die das Wechselwirkungsbild beinhalten soll. Nach dem Vorbild Heisenbergs können wir zu Gleichung (20) ganz formal einen Streuoperator $\mathbf{S}$ konstruieren: Es ist

$$|A_2\rangle = \mathbf{S}|A_1\rangle$$

genau dann, wenn eine Lösung $|A(t)\rangle$ von (20) mit

$$\lim_{t\to-\infty} |A(t)\rangle = |A_1\rangle, \qquad \lim_{t\to+\infty} |A(t)\rangle = |A_2\rangle$$

existiert. Da die physikalischen Operatoren hermitesch bezüglich der in Z herrschenden Metrik sind, ist der Operator $\mathbf{S}$ normerhaltend, d.h. unitär. Ist jedoch S ein vollständiger Satz, so wird im allgemeinen

$$\mathbf{S}Z(S) \not\subset Z(S) \tag{21}$$

sein und somit **S** aus dem Bereich der physikalischen Zustände herausführen. Der Fall (21) kann natürlich nicht vorkommen, wenn $Z(S) = Z$ ist und die Theorie keine unphysikalischen Zustände besitzt. Führen wir deshalb das in Abschnitt 6 besprochene "Ausfegen" bei der Theorie R, Z mit unphysikalischen Zuständen durch. Bezeichnet R', Z' die neu entstandene Theorie und **S'** den zugehörigen Streuoperator, so zeigt sich, daß zwischen **S** und **S'** keine allgemeingültigen (d.h. in jeder theoretisch möglichen Quantentheorie geltenden) Beziehungen bestehen.

In der Tat, zur Durchführung des Prozesses haben wir zu jeder Zeit t einen vollständigen Satz $S(t)$ mit $H(t) \in S(t)$ aufzusuchen und nach (4) und (5) die Abbildungen

$$Z(S_0) \xrightarrow{\tau_0(t)} Z(S(t)) \tag{22a}$$

zu bestimmen. Dann ist

$$H'(t) = \tau_0^{-1}(t) H(t) \tau_0(t). \tag{33b}$$

Es ist klar, daß der Zusammenhang zwischen den Lösungen von (20) und denen von

$$i\hbar \frac{d}{dt} |A'(t)\rangle = H'(t)|A'(t)\rangle \tag{20a}$$

beliebig kompliziert sein kann, wenn die $\tau_0(t)$ eine komplizierte Zeitabhängigkeit zeigen. Es erscheint deshalb sinnvoll, den Streuoperator erst nach Beseitigung der nicht-physikalischen Zustände zu bilden. Der mit (20a) gebildete Streuoperator **S'** ist nun unitär und führt physikalische Zustände wieder in solche über. Haben wir deshalb eine Quantentheorie mit nur eigentlichen Zuständen vor uns, so ist die Metrik definit und wir sind fertig.

Untersuchen wir nun den Fall, daß die Theorie auch uneigentliche physikalische Zustände beschreibt. Zur Vereinfachung der Schreibweise nehmen wir gleich an, daß Z bereits keine unphysikalischen Zustände mehr enthält. Betrachten wir zu diesem Zweck einen vollständigen Satz S, mit dessen Hilfe Anfangs- und Endzustand analysiert werden sollen. Es zeigt sich, daß S für diese Aufgabe nicht völlig beliebig gewählt werden darf, damit das Streuproblem eine sinnvolle Lösung besitzt:

Satz 7: Sei **S** der Streuoperator und S ein vollständiger Satz. Die Analyse des Zustandes vor und nach der Streuung mit Hilfe der Observablen aus S ist mit dem Streuprozeß verträglich, wenn

$$\varepsilon(S) \cdot \mathbf{S} = \mathbf{S} \cdot \varepsilon(S) \tag{23}$$

ist.

Denn aus (23) folgt, daß **S** nicht nur unitär bezüglich der in Z gegebenen (indefiniten!) Metrik $\langle A|B\rangle$ ist, sondern auch bezüglich der positiv definiten Metrik $\langle A|B\rangle_s$. Bei Beschränkung auf die Operatoren aus S (oder allgemeiner auf solche, die mit $\varepsilon(S)$ kommutieren) und bei Benutzung der positiv

definiten Metrik $\langle A|B\rangle_S$ ist deshalb alles wie in einer Quantentheorie mit nur eigentlichen Zuständen.

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PARTICLE-HOLE CONJUGATION IN THE SHELL MODEL

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Abstract: The idea of particle-hole conjugation is expounded in the language of second quantization. Results of Racah are rederived and in particular a theorem important for the classification of states of half-filled shells.

1. Introduction

The concept of particle-hole conjugation [1, 2]) in the atomic or nuclear shell seems most readily treated in the language of second quantization. This note is essentially a reconstruction in that language of ideas of Racah [2]). There is little overlap with related work of Brink and Satchler [3]) and of Visscher and Ferrell [4]).

We will consider a configuration with only one incomplete shell. In evaluating matrix elements between states of this configuration the expectation value over the filled shells is first taken, leaving an effective operator acting only within the incomplete shell; the filled shells can thereafter be ignored. Suppose first that all particles in the given shell have equal charge. Let

$$a_m^+, a_m \quad m = j, j-1, \ldots -j$$

be the creation and destruction operators of the single particle states spanning the shell. They have anticommutators

$$\begin{aligned} \{a_m, a_n\} = \{a_m^+, a_n^+\} &= 0, \\ \{a_m, a_n^+\} &= \delta_{mn}. \end{aligned} \tag{1}$$

Let $|0\rangle$ be the vacuum state. Then we define the conjugation operator by

$$C|0\rangle = a_j^+ a_{j-1}^+ \ldots a_{-j}^+|0\rangle \tag{2}$$

and

$$Ca_m^+ a_n^+ \ldots |0\rangle = (-1)^{m+j} a_{-m}(-1)^{n+j} a_{-n} \ldots C|0\rangle \tag{3}$$

for any number of different a^+. Because a_m^+ and $(-1)^{m+j}a_{-m}$ behave similarly under rotation (as spherical tensors of rank j) and because the filled shell (2) is rotationally invariant, the conjugate state has the same rotational properties as the original, i.e. C commutes with total angular momentum.

For this the reversal $m \to -m$ and the sign factor $(-1)^m$ (with the usual conventions) are essential.

Since C clearly transforms one complete orthonormal set into another it is unitary:

$$C^{-1} = C^{+}. \tag{4}$$

The definition (3) implies

$$\begin{aligned} Ca_m{}^{+}C^{-1} &= (-1)^{m+j}a_{-m}, \\ Ca_mC^{-1} &= (-1)^{m+j}a^{+}_{-m}, \end{aligned} \tag{5}$$

where the second follows, using (4), by Hermitian conjugation of the first. A rearrangement of these gives

$$\begin{aligned} C^{-1}a_mC &= (-1)^{m-j}a^{+}_{-m}, \\ C^{-1}a_m{}^{+}C &= (-1)^{m-j}a_{-m}. \end{aligned} \tag{6}$$

From (5)

$$C^2a_m{}^{+}C^{-2} = (-1)^{2j}a_m{}^{+}$$

and a short calculation using (2) and (5) gives

$$C^2|0\rangle = (-1)^{j(2j+1)}|0\rangle.$$

It follows that

$$C^2 = (-1)^{j(2j+1-2N)} \tag{7}$$

where

$$N = \sum_m a_m{}^{+}a_m$$

is the number of particles in the shell, and we have used

$$(-1)^{2Nj} = (-1)^{-2Nj}$$

(because N has integral eigenvalues). We are normally interested in eigenstates of N. Clearly these can be eigenstates of C also only when the shell is half filled, i.e. $2N = 2j+1$. For such states, from (7), $C^2 = 1$ and the eigenvalues of C are ± 1.

2. Single Body Tensor Operators

Let $T(k, q)$ be that part of an arbitrary single body tensor operator of rank k which acts within the given shell:

$$T(k, q) = \sum_{mn} a_m{}^{+}t^{q}_{mn}a_n.$$

We wish to find $\langle b'|T|a'\rangle$ in terms of $\langle b|T|a\rangle$ where $|b\rangle$ and $|a\rangle$ are states of the configuration and

$$|a'\rangle = C|a\rangle \qquad |b'\rangle = C|b\rangle$$

their conjugates. Using (4),

$$\langle b'|T|a'\rangle = \langle b|C^{+}TC|a\rangle = \langle b|C^{-1}TC|a\rangle.$$

From (6), using

$$(-1)^{n-j} = (-1)^{j-n}$$

and the fact that $m-n = q$ for non vanishing t^q_{mn},

$$\begin{aligned} C^{-1}TC &= (-1)^q \sum_{mn} t^q_{mn} a_{-m} a^+_{-n} \\ &= (-1)^q \{ - \sum_{mn} t^q_{-n-m} a_m{}^+ a_n + \sum_m t^q_{mn} \}. \end{aligned} \tag{8}$$

The last term equals the expectation value of T in the complete shell, and vanishes unless $k = 0$. For given j and k the matrix elements t^q_{mn} are proportional to the vector coupling coefficient [5])

$$(kqjm|kjjn)$$

and by a symmetry property of this,

$$\begin{aligned} (-1)^q t^q_{-n-m} &= (-1)^k (-1)^q t^{-q}_{nm} \\ &= (-1)^k \{t^{\dagger q}_{mn}\}^* \end{aligned} \tag{9}$$

where the Racah conjugate tensor $T^\dagger$ has been introduced. With the usual phase conventions the states $|a\rangle$, $|b\rangle$ (of good J^2 and J_z) are built up by sums of products of a^+ with real coefficients; on using (9) in (8) the complex conjugation can then be taken outside to give

$$\langle b'|T(k,q)|a'\rangle = -(-1)^k \langle b|T^\dagger(k,q)|a\rangle^* + \langle b|a\rangle \langle c|T(k,q)|c\rangle \tag{10}$$

where $|c\rangle$ is the filled shell.

3. Spin and Isobaric Spin

For particles with spin the above applies at once to the shells of jj-coupling theory, no restriction to integral j having been made. In LS-coupling we proceed as follows, including the trivial generalization to particles of isobaric spin τ as well as spin s. The creation operators are

$$a_m{}^+, \qquad m \equiv (m_l, m_s, m_\tau)$$

where m is now a composite index. We define the conjugation operator by

$$\begin{aligned} C|0\rangle &= \prod_m a_m{}^+ |0\rangle, \\ C a_m{}^+ C^{-1} &= (-1)^{m_l+l} (-1)^{m_s+s} (-1)^{m_\tau+\tau} a_{-m}, \end{aligned} \tag{11}$$

where $-m \equiv (-m_l, -m_s, -m_\tau)$. The a^+ are components of a tensor of rank l in orbital, s in spin, and τ in isobaric space; with the sign factors indicated, so also are the absorption operators into which they transform. C is therefore invariant under independent rotations in the three spaces, and preserves the corresponding quantum numbers. The equations, analogous to those of §1, are mostly obvious. Corresponding to (6)

$$C^2 = (-1)^{(l+s+\tau)\{(2l+1)(2s+1)(2\tau+1)-2N\}} \tag{12}$$

and once again $C^2 = 1$ for half filled shells. For single body tensor operators of rank $(k_l, k_s, k_\tau) \equiv k$ in the three spaces (10) is replaced by

$$\langle b'|T(k,q)|a'\rangle = -(-1)^{k_l+k_s+k_\tau}\langle b|T^\dagger|a\rangle^* + \langle b|a\rangle\langle c|T|c\rangle \qquad (13)$$

where $q \equiv (q_l, q_s, q_\tau)$; the last term is zero unless T is scalar in each space separately. The Racah conjugate is here defined by

$$t^{\dagger q}_{mn} = (-1)^{q_l+q_s+q_\tau}\{t^{-q}_{nm}\}^*.$$

By dropping the isobaric spin indices above, and with $s = \frac{1}{2}$, we have the formulae appropriate to the LS-coupling atomic shell model, and (13) is then the fundamental result of Racah (his equation (74)).

Let us relate the transformation of this section to that of §1. The (j, m) operators are related to the (m_l, m_s) by

$$a^+_{jm} = \sum_{m_l m_s} (lm_l sm_s|lsjm)a^+_{m_l m_s}.$$

Using (11) and a symmetry property of vector coupling coefficients, one finds

$$\begin{aligned} Ca^+_{jm}C^{-1} &= (-1)^{m+j+2(l+s-j)}\sum(lm_l sm_s|lsj-m)a^+_{m_l m_s} \\ &= (-1)^{m+j}a^+_{j-m}. \end{aligned}$$

Thus, the transformation is essentially that of §1 applied to the $j = l \pm \frac{1}{2}$ subshells simultaneously. It follows that if some linear combination of states of the LS-coupling schemes is a state of the jj-scheme, then the same property holds for the same linear combination of conjugate states. Thus, the transformation between the two schemes can be taken to be the same for a group of particles and the conjugate group of holes (cf. ref.[1]), Ch. XII).

4. A Theorem for Scalar Two Body Operators

We henceforth consider only the case (11) with $s = \tau = \frac{1}{2}$. Similar development for other cases are of course possible. The general many body operator could be reduced to products of single body operators and the discussion based on (13). However, in the particular case of interest the following procedure seems more elegant.

We introduce the partial field operators ψ by

$$\psi(x) = \sum a_m u_m(x)$$

where $m \equiv (m_l, m_s, m_\tau)$, and the u are the single particle spinor wave functions (their spinor indices, and those of ψ, are suppressed). This is just that part of the total field operator which acts on the given shell. From the anticommutation relations for the a and a^+, the only non-vanishing anticommutator for the field operators is

$$\{\psi(x), \psi^+(x')\} = \sum u_m(x)u_m{}^*(x'). \qquad (15)$$

From (4)

$$C^{-1}\psi(x)C = \sum a_m^+(-1)^{l+m_l+s+m_s+\tau+m_\tau}u_{-m}(x),$$

using the integral property of the exponent. Now, with the usual phase conventions, the coefficient of a_m^+ is equal to

$$(-1)^l K u_m^*(x), \qquad K = K_s K_\tau = K_\tau K_s$$

where K_s and K_τ are matrices, acting on spin and isobaric spin indices respectively, which can be expressed in terms of the usual Pauli matrices:

$$K_s = -i\sigma_y \qquad K_\tau = -i\tau_2. \tag{16}$$

Note that they are unitary, and, in this representation, real but antihermitian. Thus †,

$$\begin{aligned} C^{-1}\psi(x)C &= (-1)^l K\psi^+(x), \\ C^{-1}\psi^+(x)C &= (-1)^l \psi(x)K^+. \end{aligned} \tag{17}$$

Note that K_s, but not K_τ, appears in the time reversal transformation [6])

$$\psi(x) \to K_s\psi^*(x) \tag{18}$$

where * denotes ordinary complex (not Hermitian) conjugation [7]). However, we are concerned in what follows with operators invariant against rotation in isobaric space, and so, in particular, unaltered by

$$\psi \to -i\tau_2\psi = K_\tau\psi.$$

For such operators invariance under (18) is equivalent to invariance under

$$\psi(x) \to K\psi^*(x). \tag{19}$$

Consider now the general two body operator which is

a) scalar separately in space, spin, and isobaric spin,
b) invariant under time reversal, and
c) Hermitian.

It can be written

$$V = \int \psi^+\psi^+ v(x, y, x', y')\psi\psi \tag{20}$$

where the arguments of the field operators are x, y, x', y' in that order (which will be retained throughout), the integration is over all values of these, and v is a spinor matrix. Because of (b) (requiring $V \to V^*$ under (19))

$$V = \int (\psi^+K^+)(\psi^+K^+)v^*(x, y, x', y')(K\psi)(K\psi)$$

and then because of (c)

$$V = \int (\psi^+K^+)(\psi^+K^+)\tilde{v}(y', x', y, x)(K\psi)(K\psi) \tag{21}$$

† For application to *several* incomplete shells it might be desirable to obtain a more local transformation by omitting the factor $(-1)^l$ in (17) and (11). For a single shell the difference is trivial.

where $\sim$ denotes transposition of spinor indices. From (17) and (21)

$$C^{-1}VC = \int \psi\psi\tilde{v}(y', x', y, x)\psi^{+}\psi^{+}.$$

Using the commutation relations to reverse the order of operators and comparing the result with (20)

$$C^{-1}VC = V+R \tag{22}$$

where R is a residual term arising from the non-vanishing commutator (15) and is a c-number plus a one-body operator; the latter must be a multiple of N, for there is no other one-body operator invariant in all three spaces. It follows that for half filled shells, where C and N commute,

$$C^{-1}RC = R.$$

Now, V commutes with N, and therefore with C^2; therefore, multiplying (22) by C^{-1} on the left and C on the right,

$$\begin{aligned} V &= C^{-1}VC+C^{-1}RC \\ &= C^{-1}VC+R. \end{aligned} \tag{23}$$

It follows from (23) and (22) that $R = 0$, and

$$C^{-1}VC = V. \tag{24}$$

Thus, for the half filled shell V and C commute; if the former is the interaction potential, the latter provides a good quantum number for classification of states.

Equation (24) can be extended to permit tensor forces. The scalar-scalar character in space and spin was used only to ensure that R in (22) be a multiple of N plus a c-number. The only other one-body operator scalar in space and spin combined is the vector-vector. So the scalar-scalar restriction on V can be relaxed to overall scalarity *and no vector-vector part.* We should consider also the one-body force representating interaction with closed shells; if this were the appropriate average of a two-body force satisfying the above requirements, it could only be a multiple of N; in practice of course the spin-orbit force is important and satisfies (24) with opposite sign on the right hand side.

5. Conjugate Partitions

Shell model states are conventionally classified into multiplets, each of which spans an irreducible representation of the group of unitary transformations in spin space. If α is a unitary matrix acting on spinor indices a corresponding unitary transformation of state vectors can be defined by

$$U(\alpha)|0\rangle = |0\rangle, \tag{25}$$

$$U(\alpha)\psi^{+}(x)U^{-1}(\alpha) = \psi^{+}(x)\alpha. \tag{26}$$

Note that $U(\alpha)U(\beta) = U(\alpha\beta)$. The transformation induced on the conjugate state is

$$\begin{aligned} U_c(\alpha) &= CU(\alpha)C^{-1} \\ &= C^{-1}C^2U(\alpha)C^{-2}C \\ &= C^{-1}U(\alpha)C, \end{aligned} \tag{27}$$

where the last expression is valid because U commutes with N and therefore with C^2. Using (27), (11), and (25) expressed in terms of the a^+, one finds

$$U_c(\alpha)|0\rangle = |\alpha|^{2l+1}|0\rangle = |0\rangle$$

if we make the trivial restriction $|\alpha| = 1$. Moreover, from (26), (17), and the Hermitian conjugate of (25),

$$U_c^{-1}\psi^+U_c = \psi^+(K^+)^{-1}\alpha^*K^+.$$

Thus,

$$U_c(\alpha) = U\big((K^+)^{-1}\alpha^*K^+\big). \tag{28}$$

As α varies over all unitary matrices, so also does $(K^+)^{-1}\alpha^*K^+$. It follows that if a set of states forms an irreducible representation, so also does the set of conjugates. It remains only to find how the two representations are related.

A multiplet is specified by a partition of N into four integers,

$$N = \Lambda_4+\Lambda_3+\Lambda_2+\Lambda_1, \quad \Lambda_4 \geqq \Lambda_3 \geqq \Lambda_2 \geqq \Lambda_1$$

(the numbers of elements in the *columns* of the usual Young tableau) which according to Wigner [9]) can be defined as follows. The multiplet can be spanned by a set of eigenfunctions of the occupation numbers $(\mu_1, \mu_2, \mu_3, \mu_4)$ of the four spin states. Then Λ_4 is the maximum value of any μ, $\Lambda_4+\Lambda_3$ the maximum sum of any two (compatible) μ, and $\Lambda_4+\Lambda_3+\Lambda_2$ the maximum sum of any three. It follows that Λ_1 is the minimum of any μ, $\Lambda_1+\Lambda_2$ the minimum sum of any two, and so on. Now, under conjugation,

$$(\mu_1, \mu_2, \mu_3, \mu_4) \to (2l+1-\mu_1, 2l+1-\mu_2, 2l+1-\mu_3, 2l+1-\mu_4).$$

Thus,

$$(\Lambda_4, \Lambda_3, \Lambda_2, \Lambda_1) \to (2l+1-\Lambda_1, 2l+1-\Lambda_2, 2l+1-\Lambda_3, 2l+1-\Lambda_4) \tag{29}$$

is the required relation.

In the nuclear p shell there are several pairs of states which are not separated by the partition together with the rotational quantum numbers [8]). In each case the partition is self conjugate and the two states can be chosen to have eigenvalues ± 1 of C. It follows from the last section that the matrix of a spin and isobaric spin independent two-body force is then completely diagonalized. In the presence of scalar or tensor spin dependence the partition no longer provides good quantum numbers, unlike the conjugation operator; the latter, however, gives a very incomplete classification.

I am indebted to Drs. J. P. Elliott and J. M. Soper for valuable discussion of this subject.

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A PROPOSED STRONG INTERACTION OF MUONS

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Abstract: The existence of a moderately strong universal interaction between baryons-K-mesons and muons-K-mesons has been assumed. This assumption may explain the anomalously large mass of muons. The interaction further naturally gives rise to several other effects, e.g. to a short range strong singular muon-nucleon potential. These effects do not seem to contradict the experiments.

1. Introduction

Although a satisfactory theory which would explain the origin of the masses of elementary particles does not exist, it is generally believed that in the solution of this problem the interactions of the particle with other fields will play a decisive role. This opinion is confirmed by the experience that the strongly interacting particles have larger masses, the weakly interacting particles smaller ones.

According to the experimental mass-differences

$$\mathrm{e}-\nu = 0.5\ \mathrm{MeV}, \qquad \mathrm{n}-\mathrm{p} = 1.3\ \mathrm{MeV},$$
$$\pi^+-\pi^0 = 4.1\ \mathrm{MeV}, \qquad \Sigma^- - \Sigma^+ = 6\ \mathrm{MeV},$$

it seems correct to conclude that *the electromagnetic interaction leads to self-energies of order* 1 MeV.

The fermions interacting with the pion field possess masses about 1 000 MeV larger than those of the leptons:

$$\mathrm{p}-\mathrm{e} = 936\ \mathrm{MeV}, \qquad \mathrm{n}-\nu = 938\ \mathrm{MeV},$$
$$\mathrm{p}-\mu = 831\ \mathrm{MeV};$$

thus, we may suppose that *the pionic interaction gives rise to fermion self-energies of the order of* 1 000 MeV. The ratio of the pionic and electromagnetic self-energies is of the order of the ratio of the coupling constants.

The assumption that the mass-differences of the hyperons,

$$\Sigma^0-\Lambda^0 = 75\ \mathrm{MeV}, \qquad (\Lambda^0-\mathrm{n} = 175\ \mathrm{MeV}),$$
$$(\Xi^- - \Sigma^- = 124\ \mathrm{MeV}),$$

are caused by the moderately strong K-mesonic interactions (see e.g. ref.[1]))

also fits the above conception. Thus, *the fermion self-energies caused by* K-*mesonic interactions may be of the order of* 100 MeV.

It is very tempting to conclude from the above facts that particles having only the same type of interactions (with equal strength) should have equal masses. The large mass-difference of the muon and electron,

$$\mu - e = 105 \text{ MeV},$$

however, seems to be in definite contradiction with the above principle, because these two particles have exactly the same known interactions. This problem was emphasized most recently by Feynman [2]).

For the explanation of the extraordinarily large mass of the muons Schwinger suggested [3]) that these particles interact strongly with a hitherto unobserved boson, the so-called isosingulet σ-meson. Detailed calculations [4, 5]) showed, however, that for the elimination of the experimentally not observed consequences of this strong coupling one must choose a σ-mass equal to about 3 nucleon masses. The large $\sigma - \pi$ mass-difference on the other hand makes it very improbable that they form a supermultiplet as the quartet $\Lambda - \Sigma$.

The question arises: is it possible to solve the mystery of the muons by assuming a known type interaction with the already discovered particles? According to the foregoing the extra muon self-energy of 105 MeV unambiguously points to the K-mesonic interaction as its cause from among the couplings of known type. In the following we analyse the consequences of the assumption of a moderately strong muon—K-meson coupling.

The correct form of the K-mesonic interactions is not yet fully established. According to the most usual assumption the baryons differing in strangeness by one are coupled by the K-meson field (NΛK, NΣK, $\Lambda\Xi$K, $\Sigma\Xi$K, see e.g. ref. [6])). If we wished to extend this conception to the case of muons we should assume a series of particles (possessing different strangeness values similar to those of the baryon family), the muon being the lowest (and so far the only known) member of it. Although experience does not seem to be in immediate contradiction to this, since the probability of production of "muons" having strangenesses $|S| > 0$ would be small because of the lack of direct strong muon-baryon-coupling and of the large mass-difference between the members of the series, one would have to introduce several unknown features in the course of the development of the theory. Another proposal for the K-mesonic interaction is due to M. Goldhaber and Györgyi [7, 8, 9]). According to this, a direct coupling takes place between nucleon-pairs and K-meson-pairs. (The hyperons would be bound N$\overline{\text{K}}$ and N$\overline{\text{KK}}$ systems.) One can get a correct value for the masses of hyperons by choosing the value of the coupling constant ≈ 1 [9]).

In the following we assume that *the* K-*meson-field is coupled, not only to*

the baryons, but also to the muons, with the same coupling constant (fig. 1). Thus, there seems to be no need of assuming either a new particle or a new type of interaction (or a new coupling constant) for the understanding of the strikingly large mass of the muon.

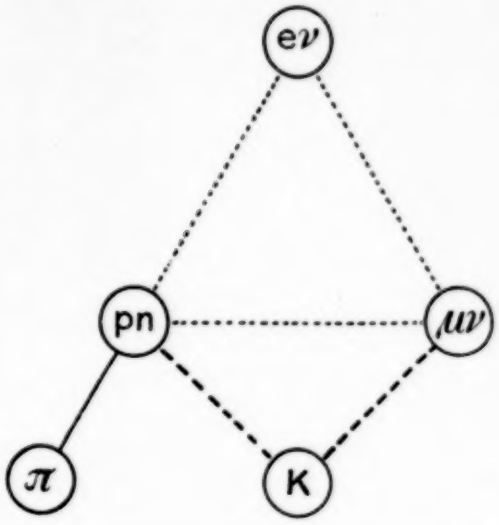

Fig. 1

2. The Strong Interaction of the Muon

We assume that the interaction Lagrangian has the form †

$$L_S = (g\bar{\psi}_\alpha \psi_\alpha + g'\bar{\chi}\chi)\phi_\beta^+ \phi_\beta \qquad (\alpha, \beta = 1, 2) \qquad (1')$$

(scalar coupling) or

$$L_V = \tfrac{1}{2}(G\bar{\psi}_\alpha \gamma_\mu \psi_\alpha + G'\bar{\chi}\gamma_\mu \chi)\phi_\beta^+ [\overleftarrow{\partial}_\mu - \overrightarrow{\partial}_\mu]\phi_\beta \qquad (\alpha, \beta = 1, 2) \qquad (1'')$$

(vector coupling); α, β are isospin indices; the nucleon is described by the isospinor ψ_α, the K-meson by the isospinor ϕ_β, the muon by the isoscalar χ; g, g' and G, G' are coupling constants having the dimension cm and cm^2, respectively. The vector coupling is preferred because this gives rise only to a $N\overline{K}$ bound system, the NK forces being repulsive [8]). In the spirit of the universality of the K-mesonic interaction we suppose $g = g'$, $G = G'$. According to experiment [1,9]) $g\mu \approx G\mu^2 \approx 1$, where μ is the inverse Compton wave length of the K-meson.

The μ—K interaction cannot be observed directly because μ—μ, μ—K, K—K collisions cannot be realized, and decay processes cannot occur via this coupling (fig. 2).

The most important consequence of (1) is a strong muon-nucleon inter-

† An isoscalar muon cannot take part in an isovector interaction.

Fig. 2

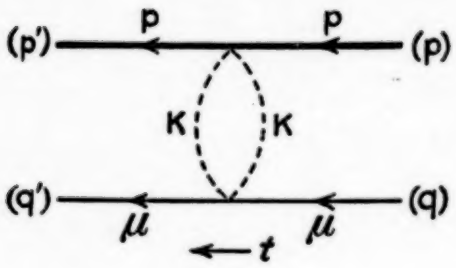

Fig. 3

action (fig. 3). The static potentials corresponding to fig. 3 are

$$V(r) = -\frac{gg'\hbar c}{2(2\pi)^3}\frac{e^{-2\mu r}}{r^3} \tag{2'}$$

for scalar coupling, and

$$V(r) = \frac{GG'\hbar c}{4(2\pi)^3}\frac{e^{-2\mu r}}{r^5} \tag{2''}$$

for vector coupling. The characteristic features of these potentials are the very short range (0.2×10^{-13} cm) but strong singularity.

One may think that the effect of this potential would manifest itself in the spectrum of muon-atoms. From (2') the K-mesonic potential for the muon in a nucleus of density $\rho = 3/4\pi r_0^3$ is found to be

$$\begin{aligned}\bar{V} &\approx -\tfrac{3}{4}m_K c^2\frac{(g\mu)^2}{(2\pi\mu r_0)^3}\log\tfrac{1}{2}\left(1+\frac{\lambda^2}{\mu^2}\right)\\ &= -\,0.1\text{ MeV}\,(g\mu)^2\log\tfrac{1}{2}\left(1+\frac{\lambda^2}{\mu^2}\right),\end{aligned} \tag{3}$$

i.e. about one per cent of the Coulomb interaction. Here λ is the regularizing mass for the K-meson field. (We obtain a similar result after regularizing (2'').) The potential (3) gives an apparent decrease of the nuclear radius of the same order of magnitude. In the case of (2'') the "equivalent" nuclear radius calculated from muon-atomic spectra increases.

An other more direct experiment is the scattering of muon on nucleons (or nuclei). From (1) in the lowest order of approximation (fig. 3) we obtain the cross section

$$\sigma = -\frac{\pi^2 g^2 g'^2 M^2 m^2}{(2\pi)^8} \int\int \frac{\left| I\left(\left[\frac{q-q'}{\mu^2}\right]^2\right)\right|^2}{\sqrt{(pq)^2 - m^2 M^2}} \left(1-\frac{pp'}{M^2}\right)\left(1-\frac{qq'}{m^2}\right) \cdot \delta(p'^2+M^2)\delta(q'^2+m^2)\delta(p+q-p'-q')\mathrm{d}p'\,\mathrm{d}q', \tag{4}$$

where $I(x)$ is a regularized integral (diverging logarithmically):

$$I(x) = \int_0^1 \log \frac{xt(1-t)+1}{xt(1-t)+1-(1-\lambda^2/\mu^2)t}\,\mathrm{d}t; \tag{5}$$

m and M are the muon and nucleon reciprocal Compton wave lengths. The differential cross section resulting from (4) in the centre of mass system is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^2 g'^2 mM}{(2\pi)^8} \left| I\left(4\frac{m^2}{\mu^2}\beta^2\gamma^2 \sin^2\tfrac{1}{2}\theta\right)\right|^2 \frac{\gamma\left(1+\frac{m^2}{M^2}\beta^2\gamma^2\sin^2\frac{1}{2}\theta\right)(1-\beta^2\cos^2\frac{1}{2}\theta)}{\sqrt{1+2\frac{m}{M}\sqrt{1+\frac{m^2}{M^2}\beta^2\gamma^2+\frac{m^2}{M^2}\gamma^2(1+\beta^2)}}}, \tag{6}$$

where $\beta = v(\mu)/c$, $\gamma^{-1} = \sqrt{(1-\beta^2)}$. For $\beta = 0$ and $\theta = 0$ we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(0,0) \approx 10^{-33}\,\mathrm{cm}^2(\mu g)^2(\mu g')^2\left(1-\frac{2\lambda^2}{\lambda^2-\mu^2}\log\frac{\lambda}{\mu}\right). \tag{7}$$

Comparing (7) with the corresponding formula for the Rutherford scattering,

$$\frac{\mathrm{d}\sigma_R}{\mathrm{d}\Omega}(0,0) = \left(\frac{e^2}{4\pi\hbar c}\right)^2 \frac{1}{m^2}\,\frac{1}{(2\beta\sin\frac{1}{2}\theta)^4},$$

we see that in the forward direction the Coulomb scattering predominates, but that at larger angles the K-mesonic scattering approaches the electromagnetic scattering. The experimental results cited by Gatland [5]) show such an anomaly for larger angles: the observed scattering of the muons is stronger than would be obtained by assuming electromagnetic interaction only.

Another consequence of the assumed interaction is the muon pair creation in high energy nuclear processes via strong interactions. One of the simplest processes of this type is the muon pair creation in $N\bar{N}$ annihilation (fig. 4).

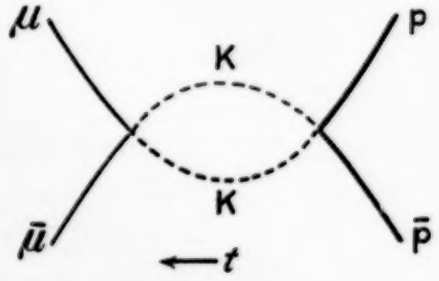

Fig. 4

The cross section is naturally similar to (6); its value is comparable to that of the photonic annihilation and is of smaller order of magnitude than the π- and K-mesonic annihilation. Similarly, it is also less probable to observe the muon pair production in nucleon-nucleon collision (fig. 5) than the emission of pions only.

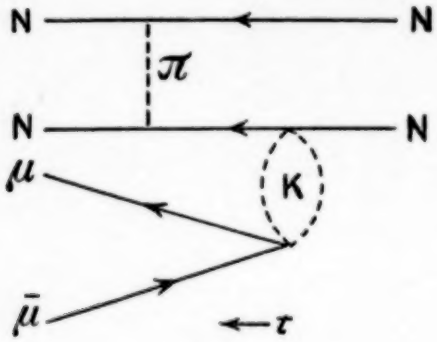

Fig. 5

Summarizing the above results, it can be said that the most sensitive phenomenon for the observation of the strong interaction of muons is the large angle scattering on nuclei. In the case of forward scattering (and, what is essentially the same, in the case of muon-atoms) the Coulomb interaction predominates as a result of its long range.

3. The Mass and Anomalous Magnetic Moment of the Muon

It seems that the consequences of the interaction treated so far have not led to effects contradicting the observations. Physically, the reason is quite clear: the interaction is strong but has a very short range. The energies occurring in muon experiments are not so high that these particles could be strongly affected by the singular domain. The self-energy, however, is a typical problem connected with very small distances. In such a small domain the singularity starts to manifest itself; we might therefore hope that the

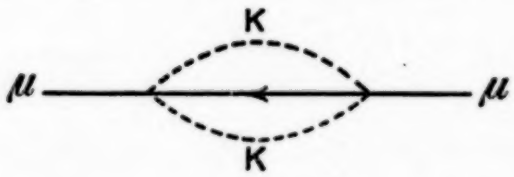

Fig. 6

interaction leads to an appropriately large self-mass. This makes it understandable that at moderate energies the predominant coupling is electromagnetic, while in the self-energy problem the K-coupling predominates †.

† P. Surányi has also tried to explain the origin of the large muon mass by assuming a strong and very short range direct muon-nucleon interaction [11].

Indeed according to fig. 6, in the lowest order of approximation one obtains the muon self-mass

$$\delta m \approx \frac{g^2}{(4\pi)^4} m\lambda^2$$

in the case of scalar and

$$\delta m \approx \frac{G^2}{(4\pi)^4} m\lambda^4$$

in case of vector coupling. Assuming that the bare muon mass is equal to the electron mass, and $m+\delta m$ is the observed muon mass, $\lambda \approx 1\,600\,\mu$ which is quite a reasonable value. (It should be noted that this cut off value is also intermediate between the corresponding electromagnetic and pionic cut off masses, as expected.)

There is one more problem we want to touch: the problem of the anomalous magnetic moment of the muon. Experimentally its value is known with a great accuracy [12]); $g_\mu = 2(1.0015 \pm 0.0006)$, which is just about the value resulting from the electromagnetic interaction. Thus, the K-mesonic interaction is expected not to give any large correction to that value. Indeed,

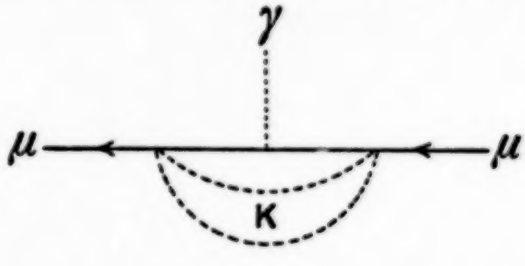

Fig. 7

calculating the anomalous magnetic moment resulting from the assumed coupling (1′) (fig. 7) we obtain

$$\tfrac{1}{2}\delta g_\mu \approx \frac{2g^2m^2}{(2\pi)^4} \log\frac{\lambda}{\mu}.$$

Substituting $\lambda = 1\,600\,\mu$, this gives

$$\tfrac{1}{2}\delta g_\mu \approx 10^{-4}$$

which is indeed small. The situation is similar in the case of vector coupling (1″): the degree of divergence becomes lower for the magnetic moment than for the self-energy.

So far in the course of the calculation we assumed the same coupling constant for the nucleon (wearing pionic clothes) and for the muon (taking part only in weaker interactions): $g = g'$, $G = G'$. If the bare coupling constants satisfy this condition, the pionic interaction does not destroy the

symmetry, because in the case of the assumed isoscalar coupling the compensation of infinities is ensured (Ward identity).

4. Conclusion

Assuming an intermediately strong N—K and μ—K interaction of the form (1) one may hope to solve the muon mystery, without facing the necessity to postulate new types of particles or interactions. The physical consequences of this coupling do not seem to contradict the known experimental facts. The best proof of the proposed muonic interaction is very likely the comparison of the above results with large angle muon nucleon scattering. In almost every step of the calculations one meets, however, divergences (the interaction not being renormalizable); thus, from that point of view the calculations do not give quite reliable results. This fact, however, does no exclude a priori an interaction of the above type, because it seems that even in the best renormalizable theories just as serious difficulties occur.

The authors hope to discuss in due course other consequences of direct muon—K-meson couplings of the above discussed or of a different type.

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